In-Class Problems Week 13, Mon.

Guess the Bigger Number Game

Team 1:
- Write two different integers between 0 and 7 on separate pieces of paper.
- Put the papers face down on a table.

Team 2:
- Turn over one paper and look at the number on it.
- Either stick with this number or switch to the other (unseen) number.

Team 2 wins if it chooses the larger number; else, Team 1 wins.

Problem 1.
The analysis given before class implies that Team 2 has a strategy that wins 4/7 of the time no matter how Team 1 plays. Can Team 2 do better? The answer is “no,” because Team 1 has a strategy that guarantees that it wins at least 3/7 of the time, no matter how Team 2 plays. Describe such a strategy for Team 1 and explain why it works.

Problem 2.
Let \( I_A \) and \( I_B \) be the indicator variables for events \( A \) and \( B \). Prove that \( I_A \) and \( I_B \) are independent iff \( A \) and \( B \) are independent.

*Hint:* Let \( A^1 := A \) and \( A^0 := \bar{A} \), so the event \([I_A = c]\) is the same as \( A^c \) for \( c \in \{0, 1\} \); likewise for \( B^1, B^0 \).

Problem 3.
Let \( R_1, R_2, \ldots, R_m \), be mutually independent random variables with uniform distribution on \([1, n]\). Let \( M := \max\{R_i \mid i \in [1, m]\} \).

(a) Write a formula for PDF\(_M\)(1).

(b) More generally, write a formula for \( \Pr[M \leq k] \).

(e) For \( k \in [1, n] \), write a formula for PDF\(_M\)(k) in terms of expressions of the form “\( \Pr[M \leq j] \)” for \( j \in [1, n] \).
Problem 4.
Suppose you have a biased coin that has probability \( p \) of flipping heads. Let \( J \) be the number of heads in \( n \) independent coin flips. So \( J \) has the general binomial distribution:

\[
PDF_J(k) = \binom{n}{k} p^k q^{n-k}
\]

where \( q := 1 - p \).

(a) Show that

\[
PDF_J(k - 1) < PDF_J(k) \quad \text{for} \quad k < np + p.
\]

\[
PDF_J(k - 1) > PDF_J(k) \quad \text{for} \quad k > np + p.
\]

(b) Conclude that the maximum value of \( PDF_J \) is asymptotically equal to

\[
\frac{1}{\sqrt{2\pi npq}}.
\]

Hint: For the asymptotic estimate, it’s ok to assume that \( np \) is an integer, so by part (a), the maximum value is \( PDF_J(np) \). Use Stirling’s Formula.

Supplemental problem

Problem 5.
You have just been married and you both want to have children. Of course, any child is a blessing, but your spouse prefers girls, so you decide to keep having children until you have a girl. In other words, if your 1st child is a girl, you’ll stop there. If it’s a boy, then you’ll have a 2nd child, too. If that one is a girl, you’ll stop there. Otherwise, you’ll have a 3rd child, and so on. Assume that you will never abandon this ingenious plan and that every child is equally likely to be a boy or a girl, independently of the number of its brothers so far. Let \( B \) be the boys that you will eventually have to put up with to enjoy the company of your beloved daughter.

(a) For \( i = 0, 1, 2, \ldots \), what is the value of \( PDF_B(i) \)?

(b) For \( i = 0, 1, 2, \ldots \), what is the value of \( CDF_B(i) \)?