In-Class Problems Week 14, Wed.

Problem 1. (a) Find a stationary distribution for the random walk graph in Figure 1.

(b) Explain why a long random walk starting at node $x$ in Figure 1 will not converge to a stationary distribution. Characterize which starting distributions will converge to the stationary one.

(c) Find a stationary distribution for the random walk graph in Figure 2.

(d) If you start at node $w$ in Figure 2 and take a (long) random walk, does the distribution over nodes ever get close to the stationary distribution? You needn’t prove anything here, just write out a few steps and see what’s happening.

(e) Explain why the random walk graph in Figure 3 has an uncountable number of stationary distributions.

(f) If you start at node $b$ in Figure 3 and take a long random walk, the probability you are at node $d$ will be close to what fraction? Explain.
(g) Give an example of a random walk graph that is not strongly connected but has a unique stationary distribution. *Hint:* There is a trivial example.

**Problem 2.**

Prove that for finite random walk graphs, the uniform distribution is stationary if and only if the probabilities of the edges coming into each vertex always sum to 1, namely

\[ \sum_{u \in \text{into}(v)} p(u, v) = 1, \]  

(1)

where \( \text{into}(w) := \{v \mid (v \rightarrow w) \text{ is an edge}\} \).

**Problem 3.**

A Google-graph is a random-walk graph such that every edge leaving any given vertex has the same probability. That is, the probability of each edge \((v \rightarrow w)\) is \(1/outdeg(v)\).

A digraph is symmetric if, whenever \((v \rightarrow w)\) is an edge, so is \((w \rightarrow v)\). Given any finite, symmetric Google-graph, let

\[ d(v) := \frac{\text{outdeg}(v)}{e}, \]

where \(e\) is the total number of edges in the graph.

(a) If \(d\) was used for webpage ranking, how could you hack this to give your page a high rank? ...and explain informally why this wouldn’t work for “real” page rank using digraphs?

(b) Show that \(d\) is a stationary distribution.