Conditional Probability

Conditional Probability: A Fair Die

\[ \Pr[\text{roll 1}] = \frac{1}{\{1,2,3,4,5,6\}} = \frac{1}{6} \]

"knowledge" changes probabilities:
\[ \Pr[\text{roll 1 knowing rolled odd}] = \frac{1}{\{1,3,5\}} = \frac{1}{3} \]

Monty Hall Probabilities

\[ \Pr[pick 1 | prize 1] \]
\[ \Pr[pick 2 | prize 3] \]
\[ \Pr[open 3 | prize 1 & pick 1] \]

Rolled odd: 1/3 1/3 1/3
Rolled even: 1/2 2/3 1/2

Prize location: 1/3 1/3 1/3
Door picked: 1/2 2/3 1/3
Door opened: 1/2 1/3 1/2
Conditional Probability

We were reasoning about conditional probability in the way we defined our probability spaces in the first place.

We were using:

Product Rule

\[ Pr[A \cap B] = Pr[A] \cdot Pr[B | A] \]

Conditional Probability

In fact, we use this reasoning to define conditional probability:

\[ Pr[B | A] \] is the probability of event B, given that event A has occurred:

\[ Pr[B | A] \ := \ \frac{Pr[A \cap B]}{Pr[A]} \]
Product Rule for 3

\[ \Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B|A] \cdot \Pr[C|A \cap B] \]

Conditioning Defines a New Space

Conditioning on \( A \) defines a new probability function \( \Pr_A \) where outcomes not in \( A \) are assigned probability zero, and outcomes in \( A \) have their probabilities raised in proportion to \( A \).
Conditioning Defines a New Space

Conditioning on $A$ defines a new probability function $\Pr_A$ where

$$\Pr_A[\omega] := 0 \quad \text{if } \omega \notin A,$$

$$\frac{\Pr[\omega]}{\Pr[A]} \quad \text{if } \omega \in A.$$ 

Now $\Pr[B|A] = \Pr_A[B]$. This implies conditional probability obeys all the rules, for example

**Conditional Difference Rule**

$$\Pr[B - C |A] = \Pr[B |A] - \Pr[B \cap C |A].$$