Great Expectations

Carnival Dice

Choose a number from 1 to 6, then roll 3 fair dice:
- win $1 for each match
- lose $1 if no match

Example: choose 5, then
- roll 2,3,4: lose $1
- roll 5,4,6: win $1
- roll 5,4,5: win $2
- roll 5,5,5: win $3

Is this a fair game?
Carnival Dice

Pr[0 fives] = \( \frac{5^3}{6^3} = \frac{125}{216} \)
Pr[1 five] = \( \frac{3}{1} \cdot \frac{5^2}{6^3} \cdot \frac{1}{6} \)
Pr[2 fives] = \( \frac{3}{2} \cdot \frac{5^3}{6^3} \cdot \frac{1}{6} \)
Pr[3 fives] = \( \frac{1}{6^3} \)

Carnival Dice

<table>
<thead>
<tr>
<th># matches</th>
<th>probability</th>
<th>$ won</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>125/216</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>75/216</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>15/216</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1/216</td>
<td>3</td>
</tr>
</tbody>
</table>

So every 216 games, expect

0 matches about 125 times
1 match about 75 times
2 matches about 15 times
3 matches about once

So on average expect to win:

\[
\frac{125 \cdot (-1) + 75 \cdot 1 + 15 \cdot 2 + 1 \cdot 3}{216} \approx -8\text{ cents}
\]
Carnival Dice

So on average expect to win:

$$\frac{125}{3} \cdot \frac{1}{3} \cdot \frac{17}{216} = -\frac{17}{216} \approx -8\text{ cents}$$

NOT fair!

Carnival Dice

You can “expect” to lose 8 cents per play. But you never actually lose 8 cents on any single play, this is just your average loss.

Expected Value

The expected value of random variable $R$ is the average value of $R$ --with values weighted by their probabilities

$$E[R] := \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$

so $E[\text{win in Carnival}] = -\frac{17}{216}$
Alternative definition

\[ E[R] = \sum_{\omega \in S} R(\omega) \cdot \Pr[\omega] \]

this form helpful in some proofs

proof of equivalence:

\[ [R = v] ::= \{ \omega \mid R(\omega) = v \} \]

so

\[ \Pr[R = v] ::= \sum_{\omega \in [R = v]} \Pr[\omega] \]

Now

\[ E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v] \]
proof of equivalence

\[ E[R] \triangleq \sum_{v \in \text{range}(R)} v \cdot \sum_{\omega \in [R=v]} \Pr[\omega] \]

Now

\[ = \sum_{v} \sum_{\omega \in [R=v]} v \cdot \Pr[\omega] \]

\[ = \sum_{\omega \in S} R(\omega) \cdot \Pr[\omega] \]
**Sums vs Integrals**

We get away with sums instead of integrals because the sample space is assumed countable:

\[ S = \{ \omega_0, \omega_1, \ldots, \omega_n, \ldots \} \]

**Rearranging Terms**

It’s safe to rearrange terms in sums because we implicitly assume that the defining sum for the expectation is absolutely convergent.

**Absolute convergence**

\[ E[R] := \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v] \]

the terms on the right could be rearranged to equal anything at all when the sum is not absolutely convergent.
Expected Value
also called
mean value, mean, or
expectation

Expectations & Averages
From a pile of graded exams, pick one at random, and let $S$ be its score.

We can estimate averages by estimating expectations of random variables.
We can estimate averages by estimating expectations of random variables based on picking random elements sampling.

For example, it is impossible for all exams to be above average (no matter what the townspeople of Lake Woebegone say):
\[ \Pr[R > E[R]] < 1 \]

On the other hand
\[ \Pr[R > E[R]] \geq 1 - \varepsilon \]
is possible for all \( \varepsilon > 0 \)
For example, almost everyone has an above average number of fingers.