Linearity of Expectation

R, S random variables, a, b constants

\[ E[aR + bS] = aE[R] + bE[S] \]

even if R, S are dependent

\[ E[aA + bB] = \sum_\omega (aA(\omega) + bB(\omega)) \cdot \Pr[\omega] = a(\sum_\omega A(\omega) \cdot \Pr[\omega]) + b(\sum_\omega B(\omega) \cdot \Pr[\omega]) = aE[A] + bE[B] \]

QED
Expectation of indicator $I_A$

$$E[I_A] := 1 \cdot \Pr[I_A=1] + 0 \cdot \Pr[I_A=0]$$

$$= \Pr[I_A=1]$$

$$= \Pr[A]$$

Expected #Heads in $n$ Flips

$H_i$ is indicator for Head on $i^{th}$ flip

$$\#H's = H_1 + H_2 + \ldots + H_n$$

Expected #Heads

$$E[\#H's] = E[H_1 + H_2 + \ldots + H_n]$$

so by linearity

$$= E[H_1] + E[H_2] + \ldots + E[H_n]$$

$$= n \cdot \Pr[\text{Head}] = np$$

Expected #hats returned

$n$ men each check their hat. Hats get scrambled so

$$\Pr[i^{th} \text{ man gets own hat back}] = \frac{1}{n}$$

How many men do we expect will get their hat back?
Expected #hats returned

$R_i$ indicates $i^{th}$ man got his hat returned.
Notice $R_i$ and $R_j$ are not independent!

Chinese Banquet
Say $n$ people sit around a spinner (a "lazy-Susan") with $n$ different dishes. Spin randomly. How many people do we expect will get same dish as initially?

Expected #hats returned

$R_i$ indicates $i^{th}$ man got his hat returned. Even so $E[\# \text{ hats returned}] = E[R_1 + R_2 + \ldots + R_n] = E[R_1] + E[R_2] + \ldots + E[R_n] = n(1/n) = 1$

Chinese Banquet
Now $R_i$ indicates $i^{th}$ person got same dish $R_i$'s are totally dependent — all 1 or all 0 but linearity still holds Expectation still is 1
Independent Product of Expectations

For independent $X, Y$

$E[X \cdot Y] = E[X] \cdot E[Y]$

proof by rearranging terms in the defining sum again

$E[XY] := \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y]$

(indep) $= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y]$

$= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y]$

$= \sum_y (y \Pr[Y=y] \sum_x \Pr[X=x])$
Independent Product of Expectations

\[ E[XY] := \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \]

\[ = \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \]

\[ = \sum_y (y \Pr[Y=y] \sum_x x \Pr[X=x]) \]

\[ = (\sum_x x \Pr[X=x])(\sum_y y \Pr[Y=y]) \]

\[ = E[X]E[Y] \]

Blunders

Don't assume product rule without independence.
Example: Say \( X \) takes positive and negative values with equal probability. So

\[ E[X] = 0 < E[X^2] \]
Blunders

Don't assume a reciprocal expectation rule. In general,

$$E\left[ \frac{X}{Y} \right] \neq \frac{E[X]}{E[Y]}$$

even with independence

Albert R Meyer, May 8, 2013