Set Theory: ZFC

Axioms of Zermelo-Frankel (ZFC) define the standard Theory of Sets

Some Axioms of Set Theory

Extensionality

\[ x \text{ and } y \text{ have the same elements} \]

iff

\[ \forall x \left[ x \in y \iff x \in z \right] \]

iff

\[ x \text{ and } y \text{ are members of the same sets} \]
Some Axioms of Set Theory

Extensionality
\[ \forall x [x \in y \iff x \in z] \]

iff
\[ \forall x [y \in x \iff z \in x] \]

Power set
Every set has a power set
\[ \forall x \exists p \forall s.s \subseteq x \iff s \in p \]

Some Axioms of Set Theory

Comprehension
If S is a set, and P(x) is a predicate of set theory, then
\[ \{x \in s \mid P(x)\} \]
is a set

Sets are Well Founded
According to ZF, the elements of a set have to be “simpler” than the set itself. In particular,
no set is a member of itself, or a member of a member...
Sets are Well Founded

Def: $x$ is $\in$-minimal in $y$

$x$ is in $y$, but no element of $x$ is in $y$

Some Axioms of Set Theory

Foundation

Every nonempty set has an $\in$-minimal element

Sets are Well Founded

Def: $x$ is $\in$-minimal in $y$

$x \in y$ AND

$[\forall z. z \in x \implies z \notin y]$

Some Axioms of Set Theory

Foundation

Every nonempty set has an $\in$-minimal element

$\forall x. [x \neq \emptyset \implies \exists y. y$ is $\in$-minimal in $x]$
Let $R ::= \{S\}$. If $S \in S$, then $R$ has no $\in$-minimal element. If it exists, it must be $S$, but $S \in R$ and $S \in S$, so $S$ is not $\in$-minimal in $R$.

Let $R ::= \{S\}$. $S \not\in S$ implies that

1. the collection of all sets is not a set, and
2. $W = \{s \in \text{Sets} \mid s \not\in s\}$

is the collection of all sets -- which is why it's not a set.