Countable Sets

A is countable iff can be listed $a_0, a_1, a_2, \ldots$.

same as $\mathbb{N}$ bij $A$ or $A$ finite

so $\mathbb{Z}^+$, $\mathbb{Z}$ countable

Binary words are countable

$\{0,1\}^*$ ::= finite binary words
list the (empty) string of length 0
list the 2 length-1 bit strings
then list the $2^2$ length-2 bit strings (in binary notation order)
then the $2^3$ length-3 bit strings

\[\vdots\]

$\mathbb{N} \times \mathbb{N}$ is countable

start with $(0,0)$
then $(0,1), (1,0)$
then $(0,2), (2,0), (1,1)$
then $(0,3), (3,0), (1,2), (2,1)$
\[\vdots\]
then all pairs with sum $n$
Lemma: A is countable iff can list A allowing repeats: \( \mathbb{N} \text{ surj } A \)

Corollary: A is countable iff \( C \text{ surj } A \) for some countable C

Rationals are countable

map \((m,n)\) to \(\frac{m}{n}\)

\(\mathbb{N} \times \mathbb{N} \text{ surj } \mathbb{Q}_{\geq 0}\)

countable so countable

Reals are uncountable

But \([0,1]^\omega\) and the real numbers \(\mathbb{R}\) are not countable: next lecture.