Integral Method for Sums

Harmonic Sums

\[ H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \]

\( H_n \) is the \( n^{th} \) Harmonic number

\[ B_n = \frac{H_n}{2} \]

Integral estimate for \( H_n \)

\[ H_n = \text{area of rectangles} \]
\[ > \text{area under } \frac{1}{x+1} = \int_0^n \frac{1}{x+1} \, dx = \int_1^{n+1} \frac{1}{x} \, dx = \ln(n+1) \]
Book stacking

for overhang 3, need $B_n \geq 3$
$H_n \geq 6$

integral bound: $\ln(n+1) \geq 6$

so ok with $n \geq \lceil e^6 - 1 \rceil = 403$ books

actually calculate $H_n$:
227 books are enough.

log$(n+1)$→∞ as $n$→∞, so overhang can be as big as desired!

CD cases over the edge

43 cases high -- top 4 cases completely off the table -- 1.8 or 1.9 case-lengths

stack of 43 CD's

see 6.042 Spring02 demo page for credits
Integral Sum Bounds

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a weakly decreasing function.

$$S := \sum_{i=1}^{n} f(i), \quad I := \int_{1}^{n} f(x) \, dx$$

$$I + f(n) \leq S \leq I + f(1)$$

Upper bound for $H_n$

$$H_n < \left( \int_{1}^{n} \frac{1}{x} \, dx \right) + 1 = 1 + \ln(n)$$
Asymptotic bound for $H_n$

$$\ln(n+1) < H_n < 1 + \ln(n)$$

$$H_n \sim \ln(n)$$

Asymptotic Equivalence

Def: $f(n) \sim g(n)$

$$\lim_{n \to \infty} \left( \frac{f(n)}{g(n)} \right) = 1$$

Asymptotic Equivalence ~

Example: $(n^2 + n) \sim n^2$

pf:

$$\lim_{n \to \infty} \frac{n^2 + n}{n^2} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = 1$$