The Logic of Propositions

Proving Validity
Instead of truth tables, can try to prove valid formulas symbolically using axioms and deduction rules.

Proving Validity
The text describes a bunch of algebraic rules to prove that propositional formulas are equivalent.

Algebra for Equivalence
for example, the distributive law

\[ P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R) \]
for example, DeMorgan’s law

\[ \text{NOT}(P \text{ AND } Q) \equiv \text{NOT}(P) \text{ OR } \text{NOT}(Q) \]

The set of rules for \( \equiv \) in the text are complete: if two formulas are \( \equiv \), these rules can prove it.

Another approach is to start with some valid formulas (axioms) and deduce more valid formulas using proof rules. Lukasiewicz’ proof system is a particularly elegant example of this idea.
A Proof System

Lukasiewicz' proof system is a particularly elegant example of this idea. It covers formulas whose only logical operators are IMPLIES (→) and NOT.

Lukasiewicz' Proof System

Axioms:
1) (¬P → P) → P
2) P → (¬P → Q)
3) (P → Q) → ((Q → R) → (P → R))

The only rule: modus ponens

Prove formulas by starting with axioms and repeatedly applying the inference rule.
To illustrate the proof system we'll do an example, which you may safely skip.

For example, to prove: $P \rightarrow P$
3rd axiom:

\[(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))\]

replace \(R\) by \(P\)

so apply modus ponens:

\[(P \rightarrow (\neg P \rightarrow P)) \rightarrow (((\neg P \rightarrow P) \rightarrow P) \rightarrow (P \rightarrow P))\]
A Lukasiewicz' Proof

so apply *modus ponens*:

\[
\text{Axiom 1)} \\
(((\overline{P} \rightarrow P) \rightarrow P) \rightarrow (P \rightarrow P))
\]

\[
(P \rightarrow P) \\
\text{QED}
\]

The 3 Axioms are all *valid* (verify by truth table). We know modus ponens is *sound*. So every provable formula is also *valid*.

Lukasiewicz is *Complete*

Conversely, every valid \((\text{NOT,} \rightarrow)\)-formulas is provable: the system is “complete”.

Not hard to verify but would take a full lecture; we omit it.
validity checking still inefficient

Algebraic & deduction proofs in general are no better than truth tables. No efficient method for verifying validity is known.