Recursive Functions

To define a function, $f$, on a recursively defined set $R$, define

- $f(b)$ explicitly for each base case $b \in R$
- $f(c(x))$ for each constructor, $c$, in terms of $x$ and $f(x)$

Recursive function on $M$

Def. tree-depth(s) for $s \in M$

$\text{td}(\lambda) ::= 0$

$\text{td}( [s]^{t} ) ::= 1 + \max\{\text{td}(s), \text{td}(t)\}$

$k^n$ — recursive function on $\mathbb{N}$

$\text{expt}(k, 0) ::= 1$

$\text{expt}(k, n+1) ::= k \cdot \text{expt}(k, n)$

--uses recursive def of $\mathbb{N}$:

- $0 \in \mathbb{N}$
- if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$
Recursive Functions

summary:
f: Data \rightarrow Values
f(b) def'd directly for base b
f(cnstr(x)) def'd using f(x), x

Length versus Depth

Lemma: |r| + 2 \leq 2^{td(r)+1}
for all r \in M

Proof by Structural Induction

Base case: [r = \lambda]
|\lambda|+2 = 0+2 = 2 = 2^{0+1} = 2^{td(\lambda)+1}
OK!

Size versus Depth

Constructor case: [r = [s]t]
by ind. hypothesis:

|s| + 2 \leq 2^{td(s)+1}
|t| + 2 \leq 2^{td(t)+1}

by def. of r:

|s| + |t| + 2 \leq 2^{td(s)+1} + 2^{td(t)+1}

\leq 2^{\max(td(s),td(t))+1}

\leq 2 \cdot 2^{\max(td(s),td(t))} + 1 \leq 2 \cdot 2^{td(r)}
def. of d(r)

= 2^{td(r)+1} QED!
positive powers of two

\[ 2 \in \text{PP2} \]
if \( x, y \in \text{PP2} \), then \( x \cdot y \in \text{PP2} \)

\[ 2, 4, 8, 16, 32, \ldots \in \text{PP2} \]

\[ \log_2 \text{ of PP2} \]

\[ \log_2(2) ::= 1 \]
\[ \log_2(x \cdot y) ::= \log_2(x) + \log_2(y) \]
for \( x, y \in \text{PP2} \)

\[ \log_2(4) = \log_2(2 \cdot 2) = 1 + 1 = 2 \]
\[ \log_2(8) = \log_2(2 \cdot 4) = \log_2(2) + \log_2(4) \]
\[ = 1 + 2 = 3 \]

\[ \log_{gy} \text{ function on PP2} \]

\[ \log_{gy}(2) ::= 1 \]
\[ \log_{gy}(x \cdot y) ::= x + \log_{gy}(y) \]
for \( x, y \in \text{PP2} \)

\[ \log_{gy}(4) = \log_{gy}(2 \cdot 2) = 2 + 1 = 3 \]
\[ \log_{gy}(8) = \log_{gy}(2 \cdot 4) = 2 + \log_{gy}(4) \]
\[ = 2 + 3 = 5 \]
\[ \log_{gy}(16) = \log_{gy}(8 \cdot 2) = 8 + \log_{gy}(2) \]
\[ = 8 + 1 = 9 \]

\[ \text{WAIT A SEC!} \]
\[ \log_{gy}(16) = \log_{gy}(2 \cdot 8) = \underline{9} \]
\[ \log_{gy}(16) = \log_{gy}(2 \cdot 8) \]
\[ = 2 + \log_{gy}(8) = 2 + 5 \]
\[ = 7 \]
ambiguous constructors
The Problem: more than one way to construct elements of PP2 from cnstrct(x,y) = x · y
    16 = cnstrct(8,2) but also
    16 = cnstrct(2,8)
ambiguous

ambiguous recursive defs
problem to watch out for:
recursive function on datum, e, is defined according to what constructor created e.
If 2 or more ways to construct e, then which definition to use?