Homework 5 additional problems

1. Heuristic suboptimal solution for Boolean LP. This exercise builds on exercises 4.15 and 5.13 in Convex Optimization, which involve the Boolean LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \preceq b \\
& \quad x_i \in \{0, 1\}, \quad i = 1, \ldots, n,
\end{align*}
\]

with optimal value \( p^* \). Let \( x^\text{rlx} \) be a solution of the LP relaxation

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \preceq b \\
& \quad 0 \preceq x \preceq 1,
\end{align*}
\]

so \( L = c^T x^\text{rlx} \) is a lower bound on \( p^* \). The relaxed solution \( x^\text{rlx} \) can also be used to guess a Boolean point \( \hat{x} \), by rounding its entries, based on a threshold \( t \in [0, 1] \):

\[
\hat{x}_i = \begin{cases} 
1 & \quad x^\text{rlx}_i \geq t \\
0 & \quad \text{otherwise,}
\end{cases}
\]

for \( i = 1, \ldots, n \). Evidently \( \hat{x} \) is Boolean (i.e., has entries in \( \{0, 1\} \)). If it is feasible for the Boolean LP, i.e., if \( A\hat{x} \preceq b \), then it can be considered a guess at a good, if not optimal, point for the Boolean LP. Its objective value, \( U = c^T \hat{x} \), is an upper bound on \( p^* \). If \( U \) and \( L \) are close, then \( \hat{x} \) is nearly optimal; specifically, \( \hat{x} \) cannot be more than \( (U - L) \)-suboptimal for the Boolean LP.

This rounding need not work; indeed, it can happen that for all threshold values, \( \hat{x} \) is infeasible. But for some problem instances, it can work well.

Of course, there are many variations on this simple scheme for (possibly) constructing a feasible, good point from \( x^\text{rlx} \).

Finally, we get to the problem. Generate problem data using

\[
\begin{align*}
\text{rand('state',0);} \\
n=100; \\
m=300; \\
A=\text{rand}(m,n); \\
b=\text{A*ones(n,1)/2;} \\
c=-\text{rand(n,1)};
\end{align*}
\]

You can think of \( x_i \) as a job we either accept or decline, and \(-c_i\) as the (positive) revenue we generate if we accept job \( i \). We can think of \( Ax \preceq b \) as a set of limits on
m resources. $A_{ij}$, which is positive, is the amount of resource $i$ consumed if we accept job $j$; $b_i$, which is positive, is the amount of resource $i$ available.

Find a solution of the relaxed LP and examine its entries. Note the associated lower bound $L$. Carry out threshold rounding for (say) 100 values of $t$, uniformly spaced over $[0, 1]$. For each value of $t$, note the objective value $c^T \hat{x}$ and the maximum constraint violation $\max_i (A \hat{x} - b)_i$. Plot the objective value and the maximum violation versus $t$. Be sure to indicate on the plot the values of $t$ for which $\hat{x}$ is feasible, and those for which it is not.

Find a value of $t$ for which $\hat{x}$ is feasible, and gives minimum objective value, and note the associated upper bound $U$. Give the gap $U - L$ between the upper bound on $p^*$ and the lower bound on $p^*$. If you define vectors obj and maxviol, you can find the upper bound as $U = \min(\text{obj} (\text{find} (\text{maxviol} <= 0)))$.

2. Three measures of the spread of a group of numbers. For $x \in \mathbf{R}^n$, we define three functions that measure the spread or width of the set of its elements (or coefficients).

The first function is the spread, defined as

$$\phi_{\text{spread}}(x) = \max_{i=1,\ldots,n} x_i - \min_{i=1,\ldots,n} x_i.$$ 

This is the width of the smallest interval that contains all the elements of $x$.

The second function is the standard deviation, defined as

$$\phi_{\text{stdev}}(x) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)^2\right)^{1/2}.$$ 

This is the statistical standard deviation of a random variable that takes the values $x_1, \ldots, x_n$, each with probability $1/n$.

The third function is the average absolute deviation from the median of the values:

$$\phi_{\text{abmed}}(x) = (1/n) \sum_{i=1}^{n} |x_i - \text{med}(x)|,$$

where $\text{med}(x)$ denotes the median of the components of $x$, defined as follows. If $n = 2k - 1$ is odd, then the median is defined as the value of middle entry when the components are sorted, i.e., $\text{med}(x) = x_{[k]}$, the $k$th largest element among the values $x_1, \ldots, x_n$. If $n = 2k$ is even, we define the median as the average of the two middle values, i.e., $\text{med}(x) = (x_{[k]} + x_{[k+1]})/2$.

Each of these functions measures the spread of the values of the entries of $x$; for example, each function is zero if and only if all components of $x$ are equal, and each function is unaffected if a constant is added to each component of $x$.

Which of these three functions is convex? For each one, either show that it is convex, or give a counterexample showing it is not convex. By a counterexample, we mean a specific $x$ and $y$ such that Jensen’s inequality fails, i.e., $\phi((x+y)/2) > (\phi(x) + \phi(y))/2$. 

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3. **Minimax rational fit to the exponential.** (See exercise 6.9 of *Convex Optimization.*) We consider the specific problem instance with data

\[ t_i = -3 + 6(i - 1)/(k - 1), \quad y_i = e^{t_i}, \quad i = 1, \ldots, k, \]

where \( k = 201. \) (In other words, the data are obtained by uniformly sampling the exponential function over the interval \([-3, 3].\)) Find a function of the form

\[ f(t) = \frac{a_0 + a_1 t + a_2 t^2}{1 + b_1 t + b_2 t^2} \]

that minimizes \( \max_{i=1, \ldots, k} |f(t_i) - y_i|. \) (We require that \( 1 + b_1 t_i + b_2 t_i^2 > 0 \) for \( i = 1, \ldots, k.\)) Find optimal values of \( a_0, \ a_1, \ a_2, \ b_1, \ b_2, \) and give the optimal objective value, computed to an accuracy of 0.001. Plot the data and the optimal rational function fit on the same plot. On a different plot, give the fitting error, \( i.e., f(t_i) - y_i. \)

*Hint.* You can use `strmcvx_status, 'Solved')`, after `cvx_end`, to check if a feasibility problem is feasible.

4. **Complex least-norm problem.** We consider the complex least \( \ell_p \)-norm problem

\[
\begin{align*}
\text{minimize} & \quad \|x\|_p \\
\text{subject to} & \quad Ax = b,
\end{align*}
\]

where \( A \in \mathbb{C}^{m \times n}, \ b \in \mathbb{C}^m, \) and the variable is \( x \in \mathbb{C}^n. \) Here \( \| \cdot \|_p \) denotes the \( \ell_p \)-norm on \( \mathbb{C}^n, \) defined as

\[
\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}
\]

for \( p \geq 1, \) and \( \|x\|_\infty = \max_{i=1, \ldots, n} |x_i|. \) We assume \( A \) is full rank, and \( m < n. \)

(a) Formulate the complex least \( \ell_2 \)-norm problem as a least \( \ell_2 \)-norm problem with real problem data and variable. *Hint.* Use \( z = (\Re x, \Im x) \in \mathbb{R}^{2n} \) as the variable.

(b) Formulate the complex least \( \ell_\infty \)-norm problem as an SOCP.

(c) Solve a random instance of both problems with \( m = 30 \) and \( n = 100. \) To generate the matrix \( A, \) you can use the Matlab command \( A = \text{randn}(m,n) + i*\text{randn}(m,n). \) Similarly, use \( b = \text{randn}(m,1) + i*\text{randn}(m,1) \) to generate the vector \( b. \) Use the Matlab command `scatter` to plot the optimal solutions of the two problems on the complex plane, and comment (briefly) on what you observe. You can solve the problems using the CVX functions `norm(x,2)` and `norm(x,inf)`, which are overloaded to handle complex arguments. To utilize this feature, you will need to declare variables to be `complex` in the `variable` statement. *(In particular, you do not have to manually form or solve the SOCP from part (b).)*