Convex optimization examples

- multi-period processor speed scheduling
- minimum time optimal control
- grasp force optimization
- optimal broadcast transmitter power allocation
- phased-array antenna beamforming
- optimal receiver location
Multi-period processor speed scheduling

- processor adjusts its speed $s_t \in [s_{\text{min}}, s_{\text{max}}]$ in each of $T$ time periods
- energy consumed in period $t$ is $\phi(s_t)$; total energy is $E = \sum_{t=1}^{T} \phi(s_t)$
- $n$ jobs
  - job $i$ available at $t = A_i$; must finish by deadline $t = D_i$
  - job $i$ requires total work $W_i \geq 0$
- $\theta_{ti} \geq 0$ is fraction of processor effort allocated to job $i$ in period $t$

\[
1^T \theta_t = 1, \quad \sum_{t=A_i}^{D_i} \theta_{ti} s_t \geq W_i
\]

- choose speeds $s_t$ and allocations $\theta_{ti}$ to minimize total energy $E$
Minimum energy processor speed scheduling

• work with variables $S_{ti} = \theta_{ti}s_t$

$$s_t = \sum_{i=1}^{n} S_{ti}, \quad \sum_{t=A_i}^{D_i} S_{ti} \geq W_i$$

• solve convex problem

$$\begin{align*}
\text{minimize} & \quad E = \sum_{t=1}^{T} \phi(s_t) \\
\text{subject to} & \quad s_{\text{min}} \leq s_t \leq s_{\text{max}}, \quad t = 1, \ldots, T \\
& \quad s_t = \sum_{i=1}^{n} S_{ti}, \quad t = 1, \ldots, T \\
& \quad \sum_{t=A_i}^{D_i} S_{ti} \geq W_i, \quad i = 1, \ldots, n
\end{align*}$$

• a convex problem when $\phi$ is convex

• can recover $\theta_t^*$ as $\theta_{ti}^* = (1/s_t^*)S_{ti}^*$
Example

- \( T = 16 \) periods, \( n = 12 \) jobs
- \( s^{\text{min}} = 1, \ s^{\text{max}} = 6, \ \phi(s_t) = s_t^2 \)
- jobs shown as bars over \([A_i, D_i]\) with area \( \propto W_i \)
Optimal and uniform schedules

- uniform schedule: \( S_{ti} = \frac{W_i}{(D_i - A_i + 1)} \); gives \( E^{\text{unif}} = 204.3 \)
- optimal schedule: \( S^*_{ti} \); gives \( E^* = 167.1 \)
Minimum-time optimal control

- linear dynamical system:

\[ x_{t+1} = Ax_t + Bu_t, \quad t = 0, 1, \ldots, K, \quad x_0 = x^{\text{init}} \]

- inputs constraints:

\[ u_{\text{min}} \leq u_t \leq u_{\text{max}}, \quad t = 0, 1, \ldots, K \]

- minimum time to reach state \( x_{\text{des}} \):

\[ f(u_0, \ldots, u_K) = \min \{ T \mid x_t = x_{\text{des}} \text{ for } T \leq t \leq K + 1 \} \]
state transfer time $f$ is quasiconvex function of $(u_0, \ldots, u_K)$:

$$f(u_0, u_1, \ldots, u_K) \leq T$$

if and only if for all $t = T, \ldots, K + 1$

$$x_t = A^t x^{\text{init}} + A^{t-1} Bu_0 + \cdots + Bu_{t-1} = x_{\text{des}}$$

$i.e.,$ sublevel sets are affine

**minimum-time optimal control problem:**

minimize $f(u_0, u_1, \ldots, u_K)$

subject to $u_{\text{min}} \leq u_t \leq u_{\text{max}}, \quad t = 0, \ldots, K$

with variables $u_0, \ldots, u_K$

a quasiconvex problem; can be solved via bisection
Minimum-time control example

- force \((u_t)_1\) moves object modeled as 3 masses (2 vibration modes)
- force \((u_t)_2\) used for active vibration suppression
- goal: move object to commanded position as quickly as possible, with

\[
|(u_t)_1| \leq 1, \quad |(u_t)_2| \leq 0.1, \quad t = 0, \ldots, K
\]
Ignoring vibration modes

- treat object as single mass; apply only $u_1$
- analytical (‘bang-bang’) solution

![Graphs showing system response](image)

- $(x(t))_3$
- $(u(t))_1$
- $(u(t))_2$
With vibration modes

- no analytical solution
- a quasiconvex problem; solved using bisection
Grasp force optimization

- choose $K$ grasping forces on object
  - resist external wrench
  - respect friction cone constraints
  - minimize maximum grasp force
- convex problem (second-order cone program):

$$\text{minimize} \quad \max_i \| f^{(i)} \|_2$$

$$\text{subject to} \quad \sum_i Q^{(i)} f^{(i)} = f^{\text{ext}}$$

$$\sum_i p^{(i)} \times (Q^{(i)} f^{(i)}) = \tau^{\text{ext}}$$

$$\mu_i f_3^{(i)} \geq \left( f_1^{(i)} + f_2^{(i)} \right)^{1/2}$$

variables $f^{(i)} \in \mathbb{R}^3, i = 1, \ldots, K$ (contact forces)
Example
Optimal broadcast transmitter power allocation

- $m$ transmitters, $mn$ receivers all at same frequency
- transmitter $i$ wants to transmit to $n$ receivers labeled $(i, j), j = 1, \ldots, n$
- $A_{ijk}$ is path gain from transmitter $k$ to receiver $(i, j)$
- $N_{ij}$ is (self) noise power of receiver $(i, j)$
- variables: transmitter powers $p_k, k = 1, \ldots, m$
at receiver \((i, j)\):

- signal power:
  \[ S_{ij} = A_{iji}p_i \]

- noise plus interference power:
  \[ I_{ij} = \sum_{k \neq i} A_{ijk}p_k + N_{ij} \]

- signal to interference/noise ratio (SINR): \( S_{ij}/I_{ij} \)

**problem:** choose \(p_i\) to maximize smallest SINR:

\[
\text{maximize } \min_{i,j} \frac{A_{iji}p_i}{\sum_{k \neq i} A_{ijk}p_k + N_{ij}}
\]

subject to \(0 \leq p_i \leq p_{\text{max}}\)

\ldots a (generalized) linear fractional program
Phased-array antenna beamforming

- omnidirectional antenna elements at positions \((x_1, y_1), \ldots, (x_n, y_n)\)

- unit plane wave incident from angle \(\theta\) induces in \(i\)th element a signal 
  
  \[ e^{j(x_i \cos \theta + y_i \sin \theta - \omega t)} \]

  \((j = \sqrt{-1}, \text{ frequency } \omega, \text{ wavelength } 2\pi)\)
• demodulate to get output $e^{j(x_i \cos \theta + y_i \sin \theta)} \in \mathbb{C}$

• linearly combine with complex weights $w_i$:

$$y(\theta) = \sum_{i=1}^{n} w_i e^{j(x_i \cos \theta + y_i \sin \theta)}$$

• $y(\theta)$ is (complex) *antenna array gain pattern*

• $|y(\theta)|$ gives sensitivity of array as function of incident angle $\theta$

• depends on design variables $\text{Re } w$, $\text{Im } w$
  (called *antenna array weights* or *shading coefficients*)

**design problem:** choose $w$ to achieve desired gain pattern
Sidelobe level minimization

make $|y(\theta)|$ small for $|\theta - \theta_{\text{tar}}| > \alpha$

($\theta_{\text{tar}}$: target direction; $2\alpha$: beamwidth)

via least-squares (discretize angles)

minimize $\sum_i |y(\theta_i)|^2$
subject to $y(\theta_{\text{tar}}) = 1$

(sum is over angles outside beam)

least-squares problem with two (real) linear equality constraints
\[ \theta_{\text{tar}} = 30^\circ \]

\[ |y(\theta)| \]

sidelobe level
**minimize sidelobe level** (discretize angles)

\[
\text{minimize} \quad \max_i |y(\theta_i)| \\
\text{subject to} \quad y(\theta_{\text{tar}}) = 1
\]

(max over angles outside beam)

can be cast as SOCP

\[
\text{minimize} \quad t \\
\text{subject to} \quad |y(\theta_i)| \leq t \\
\quad y(\theta_{\text{tar}}) = 1
\]
Extensions

convex (& quasiconvex) extensions:

• \( y(\theta_0) = 0 \) (null in direction \( \theta_0 \))

• \( w \) is real (amplitude only shading)

• \( |w_i| \leq 1 \) (attenuation only shading)

• minimize \( \sigma^2 \sum_{i=1}^{n} |w_i|^2 \) (thermal noise power in \( y \))

• minimize beamwidth given a maximum sidelobe level

nonconvex extension:

• maximize number of zero weights
Optimal receiver location

- $N$ transmitter frequencies $1, \ldots, N$
- transmitters at locations $a_i$, $b_i \in \mathbb{R}^2$ use frequency $i$
- transmitters at $a_1, a_2, \ldots, a_N$ are the wanted ones
- transmitters at $b_1, b_2, \ldots, b_N$ are interfering
- receiver at position $x \in \mathbb{R}^2$
• (signal) receiver power from $a_i$: $\|x - a_i\|_2^{-\alpha}$ ($\alpha \approx 2.1$)

• (interfering) receiver power from $b_i$: $\|x - b_i\|_2^{-\alpha}$ ($\alpha \approx 2.1$)

• worst signal to interference ratio, over all frequencies, is

$$S/I = \min_i \frac{\|x - a_i\|_2^{-\alpha}}{\|x - b_i\|_2^{-\alpha}}$$

• what receiver location $x$ maximizes $S/I$?
S/I is quasiconcave on \( \{x \mid S/I \geq 1\} \), i.e., on

\[
\{x \mid \|x - a_i\|_2 \leq \|x - b_i\|_2, \ i = 1, \ldots, N\}
\]

can use bisection; every iteration is a convex quadratic feasibility problem