Exercise 1.1  a) Given square matrices $A_1$ and $A_4$, we know that $A$ is square as well:

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & A_4 \end{pmatrix} \cdot \begin{pmatrix} A_1 & A_2 \\ 0 & I \end{pmatrix}$$

Note that

$$\det\begin{pmatrix} I & 0 \\ 0 & A_4 \end{pmatrix} = \det(I)\det(A_4) = \det(A_4),$$

which can be verified by recursively computing the principal minors. Also, by the elementary operations of rows, we have

$$\det = \begin{pmatrix} A_1 & A_2 \\ 0 & I \end{pmatrix} = \det\begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix} = \det(A_1).$$

Finally note that when $A$ and $B$ are square, we have that $\det(AB) = \det(A)\det(B)$. Thus we have

$$\det(A) = \det(A_1)\det(A_4).$$

b) Assume $A_1^{-1}$ and $A_4^{-1}$ exist. Then

$$AA^{-1} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

which yields four matrix equations:

1. $A_1B_1 + A_2B_3 = I,$
2. $A_1B_2 + A_2B_4 = 0,$
3. $A_4B_3 = 0,$
4. $A_4B_4 = I.$

From Eqn (4), $B_4 = A_4^{-1}$, with which Eqn (2) yields $B_2 = -A_1^{-1}A_2A_4^{-1}$. Also, from Eqn (3) $B_3 = 0$, with which from Eqn (1) $B_1 = A_1^{-1}$. Therefore,

$$A^{-1} = \begin{pmatrix} A_1^{-1} & -A_1^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{pmatrix}.$$
Exercise 1.2  

a)  
\[
\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} A_3 & A_4 \\ A_1 & A_2 \end{pmatrix}
\]

b) Let us find
\[
B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}
\]
such that
\[
BA = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 - A_3A_1^{-1}A_2 \end{pmatrix}
\]

The above equation implies four equations for submatrices

1. $B_1A_1 + B_2A_3 = A_1$
2. $B_1A_2 + B_2A_4 = A_2$
3. $B_3A_1 + B_4A_3 = 0$
4. $B_3A_2 + B_4A_4 = A_4 - A_3A_1^{-1}A_2$.

First two equations yield $B_1 = I$ and $B_2 = 0$. Express $B_3$ from the third equation as $B_3 = -B_4A_3A_1^{-1}$ and plug it into the fourth. After gathering the terms we get $B_4(A_4 - A_3A_1^{-1}A_2) = A_4 - A_3A_1^{-1}A_2$, which turns into identity if we set $B_4 = I$. Therefore
\[
B = \begin{pmatrix} I & 0 \\ -A_3A_1^{-1} & I \end{pmatrix}
\]

c) Using linear operations on rows we see that $\det(B) = 1$. Then, $\det(A) = \det(B)\det(A) = \det(BA) = \det(A_1)\det(A_4 - A_3A_1^{-1}A_2)$. Note that $(A_4 - A_3A_1^{-1}A_2)$ does not have to be invertible for the proof.

Exercise 1.3  

We have to prove that $\det(I - AB) = \det(I - BA)$.

Proof: Since $I$ and $I - BA$ are square,
\[
\det(I - BA) = \det\begin{pmatrix} I & 0 \\ B & I - BA \end{pmatrix} = \det\begin{pmatrix} I & A \\ B & I \end{pmatrix}\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix} = \det\begin{pmatrix} I & A \\ B & I \end{pmatrix}\det\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}
\]
yet, from Exercise 1.1, we have
\[
\det\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix} = \det(I)\det(I) = 1.
\]

Thus,
\[
\det(I - BA) = \det\begin{pmatrix} I & A \\ B & I \end{pmatrix}
\]

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Now,  
\[ \det \begin{pmatrix} I & A \\ B & I \end{pmatrix} = \det \begin{pmatrix} I - AB & 0 \\ B & I \end{pmatrix} = \det(I - AB). \]

Therefore  
\[ \det(I - BA) = \det(I - AB). \]

Note that \((I - BA)\) is a \(q \times q\) matrix while \((I - AB)\) is a \(p \times p\) matrix. Thus, when one wants to compute the determinant of \((I - AB)\) or \((I - BA)\), s/he can compare \(p\) and \(q\) to pick the product \((AB \text{ or } BA)\) with the smaller size.

b) We have to show that \((I - AB)^{-1} = A(I - BA)^{-1} \).

Proof: Assume that \((I - BA)^{-1}\) and \((I - AB)^{-1}\) exist. Then,

\[ A = A \cdot I = A(I - BA)(I - BA)^{-1} = (A - ABA)(I - BA)^{-1} = (I - AB)A(I - BA)^{-1} \]

\[ (I - AB)^{-1} A = A(I - BA)^{-1}. \]

This completes the proof.

**Exercise 1.6**  
\(a)\) The safest way to find the (element-wise) derivative is by its definition in terms of limits, i.e.

\[ \frac{d}{dt} (A(t)B(t)) = \lim_{\Delta t \to 0} \frac{A(t + \Delta t)B(t + \Delta t) - A(t)B(t)}{\Delta t} \]

We substitute first order Taylor series expansions

\[ A(t + \Delta t) = A(t) + \Delta t \frac{d}{dt} A(t) + o(\Delta t) \]

\[ B(t + \Delta t) = B(t) + \Delta t \frac{d}{dt} B(t) + o(\Delta t) \]

to obtain

\[ \frac{d}{dt} (A(t)B(t)) = \frac{1}{\Delta t} \left[ A(t) B(t) + \Delta t \frac{d}{dt} A(t) B(t) + \Delta t A(t) \frac{d}{dt} B(t) + \text{h.o.t.} - A(t)B(t) \right]. \]

Here “h.o.t.” stands for the terms

\[ \text{h.o.t.} = \left[ A(t) + \Delta t \frac{d}{dt} A(t) \right] o(\Delta t) + o(\Delta t) \left[ B(t) + \Delta t \frac{d}{dt} B(t) \right] + o(\Delta t^2), \]

a matrix quantity, where \(\lim_{\Delta t \to 0} \text{h.o.t.} / \Delta t = 0\) (verify). Reducing the expression and taking the limit, we obtain

\[ \frac{d}{dt} [A(t)B(t)] = \frac{d}{dt} A(t)B(t) + A(t) \frac{d}{dt} B(t). \]

\(b)\) For this part we write the identity \(A^{-1}(t)A(t) = I\). Taking the derivative on both sides, we have

\[ \frac{d}{dt} [A^{-1}(t)A(t)] = \frac{d}{dt} A^{-1}(t)A(t) + A^{-1}(t) \frac{d}{dt} A(t) = 0 \]
Rearranging and multiplying on the right by $A^{-1}(t)$, we obtain
\[
\frac{d}{dt} A^{-1}(t) = -A^{-1}(t) \frac{d}{dt} A(t) A^{-1}(t).
\]

Exercise 1.8 Let $X = \{g(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_M x^M \mid \alpha_i \in \mathbb{C}\}$.

a) We have to show that the set $B = \{1, x, \cdots, x^M\}$ is a basis for $X$.

Proof:

1. First, let’s show that elements in $B$ are linearly independent. It is clear that each element in $B$ cannot be written as a linear combination of each other. More formally,
\[
c_1(1) + c_1(x) + \cdots + c_M(x^M) = 0 \iff \forall i \ c_i = 0.
\]
Thus, elements of $B$ are linearly independent.

2. Then, let’s show that elements in $B$ span the space $X$. Every polynomial of order less than or equal to $M$ looks like
\[
p(x) = \sum_{i=0}^{M} \alpha_i x^i
\]
for some set of $\alpha_i$’s.

Therefore, $\{1, x, \cdots, x^M\}$ span $X$.

b) $T : X \to X$ and $T(g(x)) = \frac{d}{dx} g(x)$.

1. Show that $T$ is linear.

Proof:
\[
T(\alpha g_1(x) + \beta g_2(x)) = \frac{d}{dx} (\alpha g_1(x) + \beta g_2(x)) = \alpha \frac{d}{dx} g_1 + \beta \frac{d}{dx} g_2 = \alpha T(g_1) + \beta T(g_2).
\]
Thus, $T$ is linear.

2. $g(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_M x^M$, so
\[
T(g(x)) = \alpha_1 + 2\alpha_2 x + \cdots + M\alpha_M x^{M-1}.
\]
Thus it can be written as follows:

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 2 & \cdots & 0 \\
0 & 0 & 0 & 3 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & M \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\vdots \\
\alpha_M
\end{pmatrix}
= \begin{pmatrix}
\alpha_1 \\
2\alpha_2 \\
3\alpha_3 \\
\vdots \\
M\alpha_M \\
0
\end{pmatrix}.
The big matrix, $M$, is a matrix representation of $T$ with respect to basis $B$. The column vector in the left is a representation of $g(x)$ with respect to $B$. The column vector in the right is $T(g)$ with respect to basis $B$.

3. Since the matrix $M$ is upper triangular with zeros along diagonal (in fact $M$ is Hessenberg), the eigenvalues are all 0;

$$\lambda_i = 0 \forall i = 1, \ldots, M + 1.$$ 

4. One eigenvector of $M$ for $\lambda_1 = 0$ must satisfy $MV_1 = \lambda_1 V_1 = 0$

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is one eigenvector. Since $\lambda_i$’s are not distinct, the eigenvectors are not necessarily independent. Thus in order to compute the $M$ others, one uses the generalized eigenvector formula.
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