Problem 1

Let \((A, b, c, 0)\) be a state-space model of a LTI system, with \(A \in \mathbb{R}^{n \times n}, \ b, c' \in \mathbb{R}^n\). Assume that \(\lambda_i(A) + \lambda_j(A) \neq 0\), for all \(i, j\). Consider the equation

\[AX + XA + bc = 0;\]

show that there exists a non-singular matrix \(X\) that satisfies the equation if and only if \((A, b)\) is controllable and \((c, A)\) is observable.

Problem 2

Consider a LTI system described by the following state-space model:

\[
A = \begin{bmatrix}
-1 & 2 & 2 \\
-3 & -1 & -3 \\
3 & -2 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & -1 & 0
\end{bmatrix}, \quad D = 0.
\]

1. Construct a Kalman decomposition for this system, and compute the transfer function of the system. Is the system controllable/stabilizable, observable/detectable?

2. Design a stabilizing model-based compensator (i.e., composed of a full-state controller and an observer).

3. What is the transfer function of the compensator? Can you give a “classical” interpretation of the control law?
**Problem 3**

Consider a plant with transfer function \( G(s) = \frac{1}{s - 1} \). Find all feedback compensators \( K(s) \) such that (i) the closed-loop system is stable, and (ii) the output response to a unit step disturbance at the output is asymptotically zero. 

(Recall that, assuming the closed-loop transfer function \( T_{yd} \) is stable, then \( \lim_{t \to +\infty} y(t) = \lim_{s \to 0} sT_{yd}(s)D(s) \), where \( D(s) \) is the Laplace transform of the input signal.)

**Problem 4**

Consider the block diagram shown below. \( P \) is an uncertain SISO plant with transfer function \( P(s) = P_0(s) + W_1(s)\Delta_1(s) \), where \( W_1(s) \) is a stable transfer function, \( P(s) \) and \( P_0(s) \) have the same number of right half-plane poles, and

\[
|\text{Re}[\Delta_1(s)]| \leq \alpha, \quad |\text{Im}[\Delta_1(s)]| \leq \beta, \quad \forall s \in \mathbb{C}.
\]

The transfer function \( W_2(s) \) is a stable frequency weight.

1. Derive necessary and sufficient conditions for robust stability, i.e., such that the closed-loop system shown in the figure is externally stable for all admissible \( \Delta_1(s) \).

2. Assume \( P = P_0 \). Derive necessary and sufficient conditions for nominal performance, i.e., to ensure that \( \|y\| \leq \|d\| \), for all square-integrable disturbance inputs \( d \in \mathcal{L}_2 \).

3. Derive necessary and sufficient conditions for robust stability and performance, i.e., such that the closed loop system is stable, and \( \|y\| \leq \|d\| \), for all square-integrable disturbance inputs \( d \in \mathcal{L}_2 \), and for any admissible \( \Delta_1(s) \).

4. Can you give a graphical interpretation of these conditions?