Problem 1

Let $A \in \mathbb{C}^{n \times n}$, and $B \in \mathbb{C}^{m \times m}$. Show that $X(t) = e^{At}X(0)e^{Bt}$ is the solution to $\dot{X} = AX + XB$.

**Solution** — Recalling the definition of matrix exponential, $e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} (At)^i$, it is clear that, for any matrix $A$, $e^{At} = I$ for $t = 0$, and $\frac{d e^{At}}{dt} = A e^{At} = e^{At} A$.

Hence,
\[
\frac{d}{dt} (e^{At}X(0)e^{Bt}) = \left( \frac{d}{dt} e^{At} \right) X(0) e^{Bt} + e^{At} \left( \frac{d}{dt} X(0) \right) e^{Bt} + e^{At} X(0) \left( \frac{d}{dt} e^{Bt} \right) = A (e^{At}X(0)e^{Bt}) + 0 + (e^{At}X(0)e^{Bt}) B.
\]

Furthermore, for $t = 0$,
\[
(e^{At}X(0)e^{Bt})|_{t=0} = X(0).
\]

Hence we can conclude that the proposed function is in fact the solution to the initial-value problem under consideration.

Problem 2

Given two non-zero vectors $v, w \in \mathbb{R}^n$. Does there exist a matrix $A$ such that $v = Aw$ and

1. $\sigma_{\max}(A) = \sqrt{v^Tv/w^Tw}$?
2. $\|A\|_1 = \|v\|_\infty/\|w\|_\infty$?

Prove or disprove each case separately.

**Solution** — We have two cases:

1. The rank-one matrix $A = \frac{1}{w^Tw}vw^T$ has the required properties. Direct substitution shows that this matrix satisfies the condition $v = Aw$. Moreover, the only non-zero eigenvalue of the (rank-one) matrix $A^TA = \frac{1}{(w^Tw)^2}ww^Tv^Tv = \frac{v^Tv}{(w^Tw)^2}ww^T$ is equal to $\lambda_{\max}(A^TA) = v^Tv/w^Tw$, from which we get $\sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^TA)} = \sqrt{v^Tv/w^Tw}$. 


2. There is no such matrix in general. Consider the following counter-example. Pick, e.g., \( v = (1, 1) \), and \( w = (1, 0) \). The matrix \( A \) must be such that all elements in its first column are equal to 1, and hence \( \|A\|_1 \geq 2 > \|v\|_\infty / \|w\|_\infty = 1 \).

Problem 3

Use the projection theorem to solve the problem:

\[
\min_{x \in \mathbb{R}^n} \{ x^T Q x : Ax = b \},
\]

where \( Q \) is a positive-definite \( n \times n \) matrix, \( A \) is a \( m \times n \) real matrix, with rank \( m < n \), and \( b \) is a real \( m \)-dimensional vector. Is the solution unique?

**Solution** — (Note that \( Q \) being positive-definite implies it is self-adjoint, i.e., Hermitian.) Let \( x_0 \) be such that \( Ax_0 = b \), and consider the change of variables \( z = x - x_0 \). In the inner product space \( \mathbb{R}^n \), with inner product \( \langle u, v \rangle = u^T Q v \), it is desired to minimize \( \|x\|^2 = x^T Q x = \|z + x_0\|^2 \), subject to the constraint that \( z \) lies in the subspace \( M := \{ z \in \mathbb{R}^n : Az = 0 \} \). Using the projection theorem, we know that an optimal solution \( \hat{z} = \hat{x} - x_0 \) must be such that \( \langle \hat{z} + x_0 = \hat{x} \rangle \perp M \), i.e., \( \langle \hat{x}, y \rangle = \hat{x}^T Q y = 0 \), for all \( y \in M \). Summarizing, we know that

\[
\hat{x}^T Q y = 0, \quad \forall Ay = 0
\]

\[
Ax = b.
\]

In order to satisfy the first equation for all \( y \) such that \( Ay = 0 \), \( \hat{x} \) must be of the form \( \hat{x} = Q^{-1} A^T v \), for some \( v \in \mathbb{R}^m \). The vector \( v \) can be found using the constraint \( Ax = b \), i.e.,

\[
A \hat{x} = AQ^{-1} A^T v = b
\]

and hence

\[
v = (AQ^{-1} A^T)^{-1} b.
\]

Concluding,

\[
\hat{x} = Q^{-1} A^T (AQ^{-1} A^T)^{-1} b.
\]

Problem 4

Let \( \|A\| < 1 \). Show that \( \|(I - A)^{-1}\| \geq \frac{1}{1 + \|A\|} \).

**Solution**—First of all, for any vector \( x_0 \), with \( \|x_0\| = 1 \),

\[
\|(I - A)x_0\| \geq \|x_0\| - \|Ax_0\| \geq 1 - \|A\| > 0,
\]

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which shows that the matrix $I - A$ is invertible, i.e., there is no vector $x_0$, with $\|x_0\| = 1$, such that $(I - A)x_0 = 0$.

Furthermore, the following chain of inequalities holds:

\[
1 = \|I\| = \|(I - A)(I - A)^{-1}\| \leq \|I - A\| \cdot \|(I - A)^{-1}\| \\
\leq (\|I\| + \|A\|) \cdot \|(I - A)^{-1}\| = (1 + \|A\|) \cdot \|(I - A)^{-1}\|,
\]

and the result follows. The definition of induced norm implies that $\|I\| = 1$. The first inequality is due to the submultiplicative property of induced norms. The second inequality can be derived from the triangle inequality.

Problem 5

Consider a single-input discrete-time LTI system, described by

\[
x[k + 1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k] \\
y[k] = x[k],
\]

and the initial condition $x[0] = 0$. Given $M > 1$, what is the maximum value of $\|y[M]\|_2$ that can be attained with an input of “unit energy,” i.e., such that $u[0]^2 + u[2]^2 + \ldots + u[M - 1]^2 = 1$? What is the input that attains such value? How would your answer change if you were to double $M$, i.e., $M \leftarrow 2M$?

You can solve this problem symbolically; if you want to get numerical results, it is suggested you use matlab or similar program.

Solution — Let us define

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Then,

\[
y[1] = x[1] = Ax[0] + Bu[0] = Bu[0], \\
\ldots \\
\]

which can be written as

\[
y[M] = \begin{bmatrix} A^{M-1}B & A^{M-2}B & \ldots & B \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[M - 1] \end{bmatrix} = \Gamma_M U_M,
\]

where

where
\[ \Gamma_M = [A^{M-1}B \ A^{M-2}B \ \ldots \ B], \]
and
\[ U_M = [u[0] \ u[1] \ \ldots \ u[M-1]]^T. \]

The solution of the problem
\[
\max_{U_M} \quad \|y[M]\|_2 = \|\Gamma_M U_M\|_2 \\
\text{s.t.} \quad \|U_M\|_2 = 1
\]
is given by \( \sigma_{\text{max}}(\Gamma_M) \), and is attained for \( U_M = w_{\text{max}}(\Gamma_M) \), where \( w_{\text{max}} \) refers to the (right) singular vector associated with the maximum singular value.

Numerically, e.g., for \( M = 4 \), \( \sigma_{\text{max}}(\Gamma_4) = 4.1 \), \( U_M = [0.7661 \ 0.5452 \ 0.3243 \ 0.1035] \), and \( y[4] = [3.7129 \ 1.7391] \).

The output after \( 2M \) steps can be written as
\[ y[2M] = A^M \Gamma_M U'_M + \Gamma_M U''_M = \Gamma_{2M} U_{2M}, \]
where the matrix \( \Gamma_{2M} \) is defined as
\[ \Gamma_{2M} = [A^M \Gamma_M \ \Gamma_M]. \]

Clearly, \( \sigma_{\text{max}}(\Gamma_{2M}) \geq \sigma_{\text{max}}(\Gamma_M) \), i.e., \( \|y[2M]\|_2 \) can be made at least as large as \( \|y[M]\|_2 \), e.g., by concentrating the energy of the input in the last \( M \) steps (and setting the previous ones to zero).

**Problem 6**

Consider a physical system whose behavior is modeled, in continuous time, by the differential equation
\[ \dot{x} = Ax + Bu. \]

Assume that you have two sensors. The first sensor yields measurements \( y_1 = C_1x \) for \( t = 0, 1, 2, 3, \ldots \), and the second sensor yields measurements \( y_2 = C_2x \) for \( t = 0, 2, 4, \ldots \). Assuming that \( u(t) = u(\lfloor t \rfloor), \) for all \( t \geq 0 \), derive a discrete-time state-space model for the system.

**Solution** — This is a sample-and-hold system, commonly used as a model for computer-controlled systems. In this particular model, the two sensors have different sampling rate. Even though the system is not time invariant, the sampling strategy is periodic—and we can find a time-invariant model for the system exploiting this periodicity.

Consider the following expression for the response of a continuous-time LTI system.
\[ x(t_1) = e^{A(t_1-t_0)}x(t_0) + \int_{t_0}^{t_1} e^{A(t_1-\tau)} Bu(\tau) \, d\tau; \]

In particular, if \( t_0 \) is an integer, and \( t_1 = t_0 + 1 \),

\[ x(t_0 + 1) = e^A x(t_0) + \int_0^1 e^{A(1-\tau)} Bu(t_0) \, d\tau = A dx(t_0) + B_d u(t_0), \]

where \( A_d = e^A \), and \( B_d = \left( \int_0^1 e^{A(1-\tau)} \, d\tau \right) B \).

Define the output signal for the discrete-time model as

\[ y_d[k] = \begin{bmatrix} y_1(2k-1) \\ y_1(2k) \\ y_2(2k) \end{bmatrix}. \]

Similarly, define the input signal for the discrete-time model as

\[ u_d[k] = \begin{bmatrix} u(2k-1) \\ u(2k) \end{bmatrix}. \]

Finally, define the state vector as

\[ x_d[k] = x(2k - 1). \]

With these definitions in mind, one can write that

\[
\begin{align*}
y(2k-1) &= x(2k-1) \\
y(2k) &= x(2k) = A_d x(2k - 1) + B_d u(2k - 1) \\
x(2k+1) &= A_d^2 x(2k - 1) + A_d B_d u(2k - 1) + B_d u(2k)
\end{align*}
\]

The desired state-space model is as follows:

\[
\begin{align*}
x_d[k+1] &= A_d^2 x_d[k] + [A_d B_d \ B_d] u[k] \\
y_d[k] &= \begin{bmatrix} C_1 \\ C_1 A_d \\ C_2 A_d \end{bmatrix} x_d[k] + \begin{bmatrix} 0 & 0 \\ C_1 B_d & 0 \\ C_2 B_d & 0 \end{bmatrix} u_d[k].
\end{align*}
\]

Notice that this model is time-invariant, but is no longer strictly causal, since \( D \neq 0 \).