6.241 Dynamic Systems and Control
Lecture 9: Transfer Functions

Readings: DDV, Chapters 10, 11, 12

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Asymptotic Stability (Preview)

- We have seen that the unforced state response \((u = 0)\) of a LTI system is easily computed using the “A” matrix in the state-space model:

\[
x[k] = A^k x[0], \quad \text{or} \quad x(t) = e^{At} x(0).
\]

- A system is asymptotically stable if \(\lim_{t \to +\infty} x(t) = 0\), for all \(x_0\).

- Assume \(A\) is diagonalizable, i.e., \(V^{-1}AV = \Lambda\), and let \(r = Vx\) be the vector of model coordinates. Then,

\[
\begin{align*}
    r_i[k] &= \lambda_i^k r_i[0], \\
    r_i(t) &= e^{\lambda_i t} r_i(0),
\end{align*}
\]

- Clearly, for the system to be asymptotically stable, \(|\lambda_i| < 1\) (DT) or \(\text{Re}(\lambda_i) < 0\) (CT) for all \(i = 1, \ldots, n\).

- It turns out that this condition extends to the general (non-diagonalizable) case. More on this later in the course.
(Time-domain) Response of LTI systems — summary

Based on the discussion in previous lectures, the solution of initial value problems (i.e., the response) for LTI systems can be written in the form:

$$\begin{align*}
y[k] &= CA^k x[0] + C \sum_{i=0}^{k-1} (A^{k-i-1} Bu[i]) + Du[t] \\
or \\
y(t) &= C \exp(At)x(0) + C \int_0^t \exp(A(t - \tau)) Bu(\tau) \, d\tau + Du(t).
\end{align*}$$

However, the convolution integral (CT) and the sum in the DT equation are hard to interpret, and do not offer much insight.

In order to gain a better understanding, we will study the response to elementary inputs of a form that is

- particularly easy to analyze: the output has the same form as the input.
- very rich and descriptive: most signals/sequences can be written as linear combinations of such inputs.

Then, using the superposition principle, we will recover the response to general inputs, written as linear combinations of the “easy inputs.”
The continuous-time case: elementary inputs

- Let us choose as elementary input \( u(t) = u_0 e^{st} \), where \( s \in \mathbb{C} \) is a complex number.
- If \( s \) is real, then \( u \) is a simple exponential.
- If \( s = j\omega \) is imaginary, then the elementary input must always be accompanied by the “conjugate,” i.e.,
  \[
  u(t) + u^*(t) = u_0 e^{j\omega t} + u_0 e^{-j\omega t} = 2u_0 \cos(\omega t);
  \]
in other words, if \( s \) is imaginary, then \( u(t) = e^{st} \) must be understood as a “half” of a sinusoidal signal.
- If \( s = \sigma + j\omega \), then
  \[
  u(t) + u^*(t) = u_0(e^{\sigma t}e^{j\omega t} + u_0e^{\sigma t}e^{-j\omega t})
  = u_0(e^{\sigma t}(e^{j\omega t} + e^{-j\omega t})) = 2u_0 e^{\sigma t} \cos(\omega t),
  \]
and the input \( u \) is a “half” of a sinusoid with exponentially-changing amplitude.
Output response to elementary inputs (1/2)

- Recall that,

\[ y(t) = Ce^{At}x(0) + C \int_0^t e^{A(t-\tau)}Bu(\tau) \, d\tau + Du(t). \]

- Plug in \( u(t) = u_0e^{st} \):

\[
\begin{align*}
  y(t) &= Ce^{At}x(0) + C \int_0^t e^{A(t-\tau)}Bu_0e^{s\tau} \, d\tau + Du_0e^{st} \\
  &= Ce^{At}x(0) + C \left( \int_0^t e^{(sl-A)\tau} \, d\tau \right) e^{At}Bu_0 + Du_0e^{st}
\end{align*}
\]

- If \((sl - A)\) is invertible (i.e., \(s\) is not an eigenvalue of \(A\)), then

\[
y(t) = Ce^{At}x(0) + C(sI - A)^{-1} \left[ e^{(sl-A)t} - I \right] e^{At}Bu_0 + Du_0e^{st}.
\]
Output response to elementary inputs (2/2)

- Rearranging:

\[
y(t) = Ce^{At}x(0) - C(sI - A)^{-1}e^{At}Bu_0 + C(sI - A)^{-1}B + D \cdot u_0 e^{st}.
\]

  - Transient response
  - Steady-state response

- If the system is asymptotically stable, \( e^{At} \to 0 \), and the transient response will converge to zero.

- The steady state response can be written as:

\[
y_{ss} = G(s)e^{st}, \quad G(s) \in \mathbb{C}^{ny \times nu},
\]

where \( G(s) = C(sI - A)^{-1}B + D \) is a complex matrix.

- The function \( G : s \to G(s) \) is also known as the **transfer function**: it describes how the system transforms an input \( e^{st} \) into the output \( G(s)e^{st} \).
Laplace Transform

- The (one-sided) Laplace transform $F : \mathbb{C} \to \mathbb{C}$ of a sequence $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is defined as

$$F(s) = \int_{0}^{+\infty} f(t)e^{-st} \, dt,$$

for all $s$ such that the series converges (region of convergence).

- Given the above definition, and the previous discussion,

$$Y(s) = G(s)U(s).$$

$$U(s)e^{st} \implies Y(s)e^{st} = G(s)U(s)e^{st}$$

- Also, $G(s)$ is the Laplace transform of the “impulse” response.
The discrete-time case: elementary inputs

- Let us choose as elementary input $u[k] = u_0 z^k$, where $z \in \mathbb{C}$ is a complex number.
- If $z$ is real, then $u$ is a simple geometric sequence.
- Recall

$$y[k] = CA^k x[0] + C \sum_{i=0}^{k-1} A^{k-i-1} B u[i] + D u[k].$$

- Plug in $u[k] = u_0 z^k$, and substitute $l = k - i - 1$:

$$y[k] = CA^k x[0] + C \sum_{l=0}^{k-1} A^l B u_0 z^{k-l-1} + D u_0 z^k$$

$$= CA^k x[0] + C z^{k-1} \left( \sum_{i=0}^{k-1} (Az^{-1})^i \right) B u_0 + D u_0 z^k.$$
Matrix geometric series

Recall the formula for the sum of a geometric series:

$$
\sum_{i=0}^{k-1} m^i = \frac{1 - m^k}{1 - m}.
$$

For a matrix:

$$
\sum_{i=0}^{k-1} M^i = I + M + M^2 + \ldots M^{k-1}.
$$

i.e.,

$$
\sum_{i=0}^{k-1} M^i (I - M) = (I + M + M^2 + \ldots M^{k-1})(I - M) = I - M^k.
$$
Discrete Transfer Function

- Using the result in the previous slide, we get

\[ y[k] = CA^k x[0] + Cz^{k-1}(I - A^k z^{-k})(I - Az^{-1})^{-1}Bu_0 + Du_0 z^k \]

\[ = CA^k x[0] + C(z^k I - A^k)(zl - A)^{-1}Bu_0 + Du_0 z^k. \]

- Rearranging:

\[ y[k] = CA^k (x[0] - (zl - A)^{-1}Bu_0) + (C(zl - A)^{-1}B + D) u_0 z^k. \]

  - Transient response
  - Steady-state response

- If the system is asymptotically stable, the transient response will converge to zero.

- The steady state response can be written as:

\[ y_{ss}[k] = G(z)z^k, \quad G(z) \in \mathbb{C}, \]

where \( G(z) = C(zl - A)^{-1}B + D \) is a complex number.

- The function \( G : z \rightarrow G(z) \) is also known as the (pulse, or discrete) transfer function: it describes how the system transforms an input \( z^k \) into the output \( G(z)z^k \).
The (one-sided) z-transform $F : \mathbb{C} \rightarrow \mathbb{C}$ of a sequence $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ is defined as

$$F(z) = \sum_{k=0}^{+\infty} f[k]z^{-k},$$

for all $z$ such that the series converges (region of convergence).

Given the above definition, and the previous discussion,

$$Y(z) = G(z)U(z).$$

$$U(z)z^k \Rightarrow Y(z)z^k = G(z)U(z)z^k$$

$$Y(z) = G(z)U(z)$$

Also, $G(z)$ is the z transform of the “impulse” response, i.e., the response to the sequence $u = (1, 0, 0, \ldots)$. 
Models of continuous-time systems

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

\[
A = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 1 & 0 \\
-a_0 & -a_1 & \ldots & -a_{n-1}
\end{bmatrix}, \quad
B = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
b_0 & b_1 & \ldots & b_{n-1}
\end{bmatrix}, \quad
D = d
\]

\[
G(s) = C(sI - A)^{-1}B + D
\]

\[
G(s) = \frac{b_{n-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + d
\]
Models of discrete-time systems

\[
\begin{align*}
x[k + 1] &= Ax[k] + Bu[k] \\
y[k] &= Cx[k] + Du[k]
\end{align*}
\]

\[
A = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 1 & 0 \\
-a_0 & -a_1 & \cdots & -a_{n-1}
\end{bmatrix},
B = \begin{bmatrix}
0 \\
\cdots \\
0 \\
1
\end{bmatrix}
\]

\[
C = [b_0 \ b_1 \ \cdots \ b_{n-1}],
D = d
\]

\[
G(z) = C(zI - A)^{-1}B + D
\]

\[
G(z) = \frac{b_{n-1}z^{n-1} + \ldots + b_0}{z^n + a_{n-1}z^{n-1} + \ldots + a_0} + d
\]