6.241 Dynamic Systems and Control
Lecture 20: Reachability and Observability
Readings: DDV, Chapters 23, 24

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Reachability in continuous time

- Given a system described by the \((n\text{-dimensional})\) state-space model
  \[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \]
a point \(x_d\) is said to be **reachable** in time \(L\) if there exists an input \(u: t \in [0, L] \mapsto u(t)\) such that \(x(L) = x_d\).
- Given an input signal over \([0, L]\), one can compute
  \[
  x(L) = \int_0^L e^{A(L-t)}Bu(t) \, dt = \int_0^L F^T(t)u(t) \, dt =: \langle F, u \rangle_L,
  \]
  where \(F^T(t) := e^{A(L-t)}B\).
- The set \(R\) of all reachable points is a linear (sub)space: if \(x_a\) and \(x_b\) are reachable, so is \(\alpha x_a + \beta x_b\).
- If the reachable set is the entire state space, i.e., if \(R = \mathbb{R}^n\), then the system is called **reachable**.
Reachability Gramian

**Theorem**

Let $\mathcal{P}_L := \langle F, F \rangle = \int_0^L F^T(t)F(t) \, dt$. Then,

$$\mathcal{R} = \text{Ra}(\mathcal{P}_L).$$

- Prove that $\mathcal{R} \subseteq \text{Ra}(\mathcal{P}_L)$, i.e., $\mathcal{R} \perp \supseteq \text{Ra}^\perp(\mathcal{P}_L)$.

  - $q^T \mathcal{P}_L = 0 \Rightarrow q^T \mathcal{P}_L q = 0 \iff \langle Fq, Fq \rangle = 0 \iff q^T F^T(t) = 0 \Rightarrow q^T x(L) = 0$ (i.e., if $q \in \text{Ra}^\perp(\mathcal{P}_L)$, then $q \in \mathcal{R}^\perp$.)

- Now prove that $\mathcal{R} \supseteq \text{Ra}(\mathcal{P}_L)$: let $\alpha$ be such that $x_d = \mathcal{P}_L \alpha$, and pick $u(t) = F(t)\alpha$. Then

  $$x(L) = \int_0^L F^T(t)F(t)\alpha \, dt = \mathcal{P}_L \alpha = x_d.$$
Theorem

\[
Ra(\mathcal{P}_L) = Ra \left( [A^{n-1}B| \ldots |AB|B] \right) = Ra(R_m).
\]

- \(q^T \mathcal{P}_L = 0 \Rightarrow q^T e^{A(L-t)}B = 0 \Rightarrow q^T R_n = 0\)
- \(q^T R_n \iff q^T A^lB = 0, \forall l \in \mathbb{N} \Rightarrow q^T(t)e^{A(L-t)}B = 0 \forall t \in \mathbb{R} \Rightarrow q^T \mathcal{P}_L = 0.\)
- The system is reachable iff the rank of \(R_n\) is equal to \(n\).
- Notice that this condition does not depend on \(L\)!

**Reachability vs. Controllability:** a state \(x_d\) is controllable if one can find a control input \(u\) such that

\[
e^{AL}x_d + \langle F, u \rangle_L = 0.
\]

This is equivalent to \(x_d = e^{-AL} \langle F, u \rangle_L\), i.e., controllability and reachability coincide for CT systems. (They do not coincide for DT systems, e.g., is the matrix \(A\) is not invertible.)
Computation of the reachability Gramian

- Recall the definition of the reachability Gramian at time $L$:

$$P_L := \int_0^L e^{A(L-t)} BB^T e^{A(L-t)^T} dt = \int_0^L e^{A\tau} BB^T e^{A^T \tau} d\tau$$

- Recall that the range does not depend on $L$. In particular, assuming the system is stable (i.e., all eigenvalues of $A$ are in the open left half plane), one can consider $L \to +\infty$, and define

$$P := \lim_{L \to +\infty} P_L = \int_0^{+\infty} e^{A\tau} BB^T e^{A^T \tau} d\tau$$
Computation of the reachability Gramian

**Theorem**

The reachability Gramian satisfies the Lyapunov equation

\[ A\mathcal{P} + \mathcal{P}A^T = -BB^T. \]

- \[ \int_0^\infty \frac{d}{dt} \left( e^{At}BB^Te^{A^Tt} \right) \, dt = -BB^T \]
- \[ \int_0^\infty \frac{d}{dt} \left( e^{At}BB^Te^{A^Tt} \right) \, dt = A\mathcal{P} + \mathcal{P}A^T. \]
Canonical forms

- Consider the similarity transformation $AR = R\bar{A}$, $b = R\bar{b}$.

- Using Cayley-Hamilton, we get

$$\bar{A} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \ldots \\ -a_{n-2} & 0 & 1 & \ldots \\ \vdots & \ddots & \ddots & \ddots \\ -a_0 & 0 & 0 & \ldots \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 0 \\ 0 \\ \ddots \\ 1 \end{bmatrix},$$

called the controllability form.

- A similar transformation can be used to take the system to the controller canonical form...
Observability

- Given a system described by the \((n\text{-dimensional})\) state-space model
  \[
  \dot{x}(t) = Ax(t) + Bu(t)
  \]
  \[
  y(t) = Cx(t) + Du(t),
  \]
a state \(x_q \neq 0\) is said **unobservable** over \([0, L]\), if for every input \(u\), the output \(y_q\) obtained with initial condition \(x(0) = x_q\) is the same as the output \(y_0\) obtained with initial condition \(x(0) = 0\).

- A dynamic system is said **unobservable** if it contains at least an unobservable state, **observable** otherwise.

- Note that observability can be established assuming zero input.
Continuous-time observability

For continuous-time systems, the following are equivalent:

- \( x_q \) is unobservable in time \( L \)
- \( x_q \) is unobservable in any time.

\[
O_n x_q = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix} x_q = 0.
\]

1) \( \Rightarrow \) 2): if \( x_q \) unobservable in time \( L \), then for \( u = 0 \) \( y = 0 \), i.e., \( Ce^{At} x_q = 0 \) for all \( t \in [0, L] \). hence, \( Ce^{A \cdot 0} x_q = 0 \), \( d/dt Ce^{At} x_q \bigg|_{t=0} = CA x_q = 0 \) which implies that \( Ce^{At} x_q = 0 \) for all \( t > L \) as well.

2) \( \Rightarrow \) 1): immediate

2) \( \Leftrightarrow \) 3): Cayley-Hamilton implies that \( \text{Null} O_n = \text{Null} \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{l-1}
\end{bmatrix} \) for all \( l > n \).
Observability Gramian

- Define the observability Gramian at time $L$:

\[ Q_L := \int_0^L e^{A^T(L-t)} C^T C e^{A(L-t)} \, dt = \int_0^L e^{A^T \tau} C^T C e^{A \tau} \, d\tau \]

- Recall that observability does not depend on $L$. In particular, assuming the system is stable (i.e., all eigenvalues of $A$ are in the open left half plane), one can consider $L \to +\infty$, and define

\[ Q := \lim_{L \to +\infty} Q_L = \int_0^{+\infty} e^{A^T \tau} C^T C e^{A \tau} \, d\tau \]

Theorem

The observability Gramian satisfies the Lyapunov equation

\[ A^T Q + QA = -C^T C. \]
Essentially, observability results are similar to their reachability counterparts, when considering \((A^T, C^T)\) as opposed to \((A, B)\). In particular,

\((A, C)\) is unobservable if \(Cv_i = 0\) for some (right) eigenvector \(v_i\) of \(A\).

\[
y(t) = C \sum_i 1^n e^{\lambda_i t} v_i w_i^T x(0)
\]

\((A, C)\) is unobservable if \(\begin{bmatrix} sl - A \\ C \end{bmatrix}\) drop ranks for some \(s = \lambda\), this \(\lambda\) is an unobservable eigenvalue for the system.

The dual of controllability to the origin is “constructability”: same considerations as in the reachability/controllability case hold.
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