6.241 Dynamic Systems and Control
Lecture 25: $\mathcal{H}_\infty$ Synthesis

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Standard setup

Consider the following system, for $t \in R_{\geq 0}$:

$$
\dot{x}(t) = Ax(t) + B_w w(t) + B_u u(t), \quad x(0) = x_0
$$

$$
z(t) = C_z x(t) + D_{zw} w(t) + D_{zu} u(t)
$$

$$
y(t) = C_y x(t) + D_{yw} w(t) + D_{yu} u(t),
$$

where

- $w$ is an exogenous disturbance input (also reference, noise, etc.)
- $u$ is a control input, computed by the controller $K$
- $z$ is the performance output. This is a “virtual” output used only for design.
- $y$ is the measured output. This is what is available to the controller $K$

It is desired to synthesize a controller $K$ (itself a dynamical system), with input $y$ and output $u$, such that the closed loop is stabilized, and the performance output is minimized, given a class of disturbance inputs.

In particular, we will look at controller synthesis with $H_2$ and $H_\infty$ criteria.
In principle, we would like to find a controller $K$ such that minimizes the energy ($L_2$) gain of the closed-loop system, i.e., that minimizes

$$\| T_{zw} \|_{\mathcal{H}_\infty} = \sup_{w \neq 0} \frac{\|Z\|_{L_2}}{\|W\|_{L_2}}.$$

However, the optimal controller(s) are such that $\sigma_{\max}(T_{zw}(j\omega))$ is a constant over all frequencies, the response does not roll off at high frequencies, and the controller is not strictly proper. (The optimal controller is not unique.)

In addition, computing an optimal controller is numerically challenging.
A better approach in practice is to pursue a sub-optimal design, i.e., given $\gamma > 0$, find a controller $K$ such that $\| T_{zw} \|_{\mathcal{H}_\infty} < \gamma$, if one exists.

In other words, assume that the controller $K$ and the disturbance $w$ are playing a zero-sum game, in which the cost is

$$\| z \|_{L^2}^2 - \gamma^2 \| w \|_{L^2}^2.$$ 

what is the smallest $\gamma$ such that the controller can win the game (i.e., achieve a negative cost)?
Approximately optimal $\mathcal{H}_\infty$ synthesis

- The optimal performance $\gamma^*$ can be approximated arbitrarily well, by a bisection method, maintaining lower and upper bounds $\gamma_- < \gamma^* < \gamma_+$:

  - Init to, e.g., $\gamma_- = 0$, $\gamma_+ = \text{the } \mathcal{H}_\infty \text{ norm of the } \mathcal{H}_2 \text{ optimal design}$. Let $K_+$ be the optimal $\mathcal{H}_2$ controller.

  - Let $\gamma \leftarrow (\gamma_- + \gamma_+) / 2$. Check whether a controller exists such that $\|T_{zw}\|_{\mathcal{H}_\infty} < \gamma$.

  - If yes, set $\gamma_+ \leftarrow \gamma$, and set $K_+$ to the controller just designed. Otherwise, set $\gamma_- \leftarrow \gamma$.

  - Repeat from step 2 until $\gamma_+ - \gamma_- < \epsilon$.

  - Return $K_+$. 
Simplified setup

For simplicity, consider the case in which

- $C'_z D_{zu} = 0$, i.e., the cost is of the form $\int_0^{+\infty} x' Q x + u' R u \, dt$.

- $B_w D'_y = 0$, i.e., process noise and sensor noise are uncorrelated.

- $D'_{zu} D_{zu} = I$, $D_w D'_y = I$. 
Assume that the state $x$ and the disturbance $w$ are available for measurement, i.e., $y = [x \ w]'$.

Assume that the optimal control is of the form $u = F_u x$, and that the optimal disturbance is of the form $w = F_w x$.

The evolution of the system is completely determined by the initial condition $x_0$. In particular, defining $A_\infty = A + B_w F_w + B_u F_u$:

- the energy of the performance output is computed as

$$\|z\|_{\mathcal{L}_2}^2 = \int_0^{+\infty} x_0' \left( e^{A_\infty t} C_z C_z e^{A_\infty t} + e^{A_\infty t} F_u F_u e^{A_\infty t} \right) x_0 \ dt$$

- The energy of the disturbance is computed as

$$\|w\|_{\mathcal{L}_2}^2 = \int_0^{+\infty} x_0' e^{A_\infty t} F_w F_w e^{A_\infty t} x_0 \ dt.$$
Hence the cost of the game is

\[ \|z\|_{L_2}^2 - \gamma^2\|w\|_{L_2}^2 = x_0'X_\infty x_0, \]

where \(X_\infty\) is the observability Gramian of the pair \((C_\infty, A_\infty)\), with
\[ C_\infty = \begin{bmatrix} Q^{1/2} & j\gamma F'_w & F'_u \end{bmatrix}'. \]

From the properties of the observability Gramian, it must be the case that

\[ A'_\infty X_\infty + X_\infty A_\infty + C'_\infty C_\infty = 0 \]

Assuming that there exist \(S_u, S_w\) such that \(F_u = S_uX_\infty\) and \(F_w = S_wX_\infty\), and expanding, we get

\[
A'X_\infty + X_\infty S'_w B'_w X_\infty + X_\infty S'_u B'_u X_\infty \\
+ X_\infty A + X_\infty B_w S_w X_\infty + X_\infty B_u S_u X_\infty \\
+ Q - \gamma^2 X_\infty S'_w S_w X_\infty + X_\infty S'_u S_u X_\infty = 0
\]
Guess for the structure of the suboptimal controller

- A possible solution would be:

\[ A'X_\infty + X_\infty A + C_z' C_z = X_\infty (B_u B_u' - \gamma^{-2} B_w B_w') X_\infty, \]

\[ F_u = -B_u' X_\infty, \quad F_w = \frac{1}{\gamma^2} B_w' X_\infty \]

- This is a Riccati equation, but notice that the quadratic term is not necessarily sign definite.

- Similar considerations hold for the “observer” Riccati equation

\[ AY_\infty + Y_\infty A' + B'_w B_w = Y_\infty (C_y C_y' - \gamma^{-2} C_z C_z') Y_\infty \]

- The observer gain would be

\[ L = -(I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty C_y'. \]

Note the inversion of the matrix \( \gamma^2 I - Y_\infty X_\infty \).
Suboptimal $\mathcal{H}_\infty$ controller

- Assuming the following technical conditions hold:
  - $(A, B_u)$ stabilizable, $(C_y, A)$ detectable.
  - The matrices $[A - j\omega I \ B_w]$, $[A' - j\omega I \ C_z']$ must have full row rank.
- A controller $K$ such that $\|z\|_{L_2}^2 - \gamma^2 \|w\|_{L_2}^2 < 0$ exists if and only if
  - The following Riccati equation has a stabilizing solution $X_\infty \geq 0$:
    $$A'X_\infty + X_\infty A + C_z'C_z = X_\infty (B_uB_u' - \gamma^{-2}B_wB_w')X_\infty,$$
  - The following Riccati equation has a stabilizing solution $Y_\infty \geq 0$:
    $$AY_\infty + Y_\infty A' + B_w'B_w = Y_\infty (C_yC_y' - \gamma^{-2}C_zC_z')Y_\infty,$$
  - The matrix $\gamma^2I - Y_\infty X_\infty$ is positive definite.
Current research

- Distributed control systems
- Networked control systems (quantization, bandwidth limitations, etc.)
- Computational methods
- Nonlinear systems/robustness (ISS, IQCs, polynomial systems, SoS, etc.)
- Hybrid/switched systems
- System ID/Model reduction
- Robust/Adaptive control
Other classes

- 6.231 Dynamic Programming and Stochastic Control
- 6.242 Advanced Linear Control Systems
- 6.243 Dynamics of Nonlinear Systems
- 6.245 Multivariable Control Systems
- 6.256 Algebraic Techniques and Semidefinite Optimization
- 6.246-7 Advanced Topics in Control
- 2.152 Nonlinear Control System Design
- 10.552 Advanced Systems Engineering (R. Braatz on LMIs for optimal/robust control)
- 16.322 Stochastic Estimation and Control
- 16.323 Principles of Optimal Control
- 16.333 Aircraft Stability and Control