Problem Set 2\(^1\)

Problem 2.1

Consider the feedback system with external input \( r = r(t) \), a causal linear time invariant forward loop system \( G \) with input \( u = u(t) \), output \( v = v(t) \), and impulse response \( g(t) = 0.1 \delta(t) + (t + a)^{-1/2}e^{-t} \), where \( a \geq 0 \) is a parameter, and a memoryless nonlinear feedback loop \( u(t) = r(t) + \phi(v(t)) \), where \( \phi(y) = \sin(y) \). It is customary to require well-posedness of such feedback models, which will usually mean existence and uniqueness of solutions \( v = v(t), u = u(t) \) of system equations

\[
 v(t) = 0.1u(t) + \int_0^t h(t - \tau)u(\tau)d\tau, \quad u(t) = r(t) + \phi(v(t))
\]

on the time interval \( t \in [0, \infty) \) for every bounded input signal \( r = r(t) \).

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(a) Show how Theorem 3.1 from the lecture notes can be used to prove well-posedness in the case when \( a > 0 \). **Hint:** it may be a good idea to begin with getting rid of the algebraic part of the system equations by introducing a new signal \( e(t) = v(t) - 0.1\phi(v(t)) - 0.1r(t) \).

(b) Propose a generalization of Theorem 3.1 which can be applied when \( a = 0 \) as well. (You are not required to write down the proof of your generalization, but make every effort to ensure the statement is correct.)

**Problem 2.2**

Read the section of Lecture 4 handouts on limit sets of trajectories of ODE (it was not covered in the classroom).

(a) Give an example of a continuously differentiable function \( a : \mathbb{R}^2 \mapsto \mathbb{R}^2 \), and a solution of ODE

\[
\dot{x}(t) = a(x(t)),
\]

for which the limit set consists of a single trajectory of a non-periodic and non-equilibrium solution of (2.1).

(b) Give an example of a continuously differentiable function \( a : \mathbb{R}^n \mapsto \mathbb{R}^n \), and a bounded solution of ODE (2.1), for which the limit set contains no equilibria and no trajectories of periodic solutions. **Hint:** it is possible to do this with a 4th order linear time-invariant system with purely imaginary poles.

(c) Use Theorem 4.3 from the lecture notes to derive the Poincare-Bendixon theorem: if a set \( X \subset \mathbb{R}^2 \) is compact (i.e. closed and bounded), positively invariant for system (2.1) (i.e. \( x(t, \bar{x}) \in X \) for all \( t \geq 0 \) and \( \bar{x} \in X \), and contains no equilibria, then the limit set of every solution starting in \( X \) is a closed orbit (i.e. the trajectory of a periodic solution). Assume that \( a : \mathbb{R}^2 \mapsto \mathbb{R}^2 \) is continuously differentiable.

**Problem 2.3**

Use the index theory to prove the following statements.

(a) If \( n > 1 \) is even and \( F : S^n \mapsto S^n \) is continuous then there exists \( x \in S^n \) such that \( x = F(x) \) or \( x = -F(x) \).

(b) The equations for the harmonically forced nonlinear oscillator

\[
\ddot{y}(t) + \dot{y}(t) + (1 + y(t)^2)y(t) = 100\cos(t)
\]

have at least one 2\( \pi \)-periodic solution. **Hint:** Show first that, for

\[
V(t) = \dot{y}(t)^2 + y(t)^2 + y(t)\dot{y}(t) + 0.5y(t)^2,
\]
the inequality

\[ \dot{V}(t) \leq -c_1 V(t) + c_2, \]

where \( c_1, c_2 \) are some positive constants, holds for all \( t \).