Problem Set 3

Problem 3.1

Find out which of the functions $V : \mathbb{R}^2 \to \mathbb{R}$,

(a) $V(x_1, x_2) = x_1^2 + x_2^2$;
(b) $V(x_1, x_2) = |x_1| + |x_2|$;
(c) $V(x_1, x_2) = \max |x_1|, |x_2|$;

are valid Lyapunov functions for the systems

(1) $\dot{x}_1 = -x_1 + (x_1 + x_2)^3, \dot{x}_2 = -x_2 - (x_1 + x_2)^3$;
(2) $\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2), \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$;
(3) $\dot{x}_1 = x_2|x_1|, \dot{x}_2 = -x_1|x_2|$.

Problem 3.2

Show that the following statement is not true. Formulate and prove a correct version: if $V : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable functional and $a : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that

$$\nabla V(\bar{x})a(\bar{x}) \leq 0 \quad \forall \bar{x} : V(\bar{x}) = 1, \quad (3.1)$$

then $V(x(t)) \leq 1$ for every solution $x : [0, \infty) \to \mathbb{R}^n$ of

$$\dot{x}(t) = a(x(t)) \quad (3.2)$$

with $V(x(0)) \leq 1$.

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Problem 3.3

The optimal minimal-time controller for the double integrator system with bounded control
\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= u(t),
\end{align*}
\]
has the form
\[
u(t) = \text{sgn}(x_1(t) + 0.5x_2(t)^2\text{sgn}(x_2(t))).
\]
(Do you know why?)

(a) Find a Lyapunov function \( V : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) for the closed loop system, such that \( V(x(t)) \) is strictly decreasing along all solutions of system equations except the equilibrium solution \( x(t) \equiv 0 \).

(b) Find out whether the equilibrium remains asymptotically stable when the same controller is used for the perturbed system
\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -\epsilon x_1(t) + u(t),
\end{align*}
\]
where \( \epsilon > 0 \) is small.