Problem Set 4\(^1\)

Problem 4.1
Find a function \( V : \mathbb{R}^3 \mapsto \mathbb{R}_+ \) which has a unique minimum at \( \bar{x} = 0 \), and is strictly monotonically decreasing along all non-equilibrium trajectories of system
\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) + x_2(t)^2, \\
\dot{x}_2(t) &= -x_2(t)^3 + x_3(t)^4, \\
\dot{x}_3(t) &= -x_3(t)^5.
\end{align*}
\]

Problem 4.2
System \( \Delta \) takes arbitrary continuous input signals \( v : [0, \infty) \mapsto \mathbb{R} \) and produces continuous outputs \( w : [0, \infty) \mapsto \mathbb{R} \) in such a way that the series connection of \( \Delta \) and the LTI system with transfer function \( G_0(s) = 1/(s + 1) \), described by equations
\[
\begin{align*}
\dot{x}_0(t) &= -x_0(t) + w(t), \quad w(\cdot) = \Delta(v(\cdot)),
\end{align*}
\]
has a non-negative storage function with supply rate
\[
\sigma_0(\bar{x}_0, \bar{v}, \bar{w}) = (\bar{w} - 0.9\bar{x}_0)(\bar{v} - \bar{w}).
\]
(a) Find at least one nonlinear system \( \Delta \) which fits the description.
(b) Derive constraints to be imposed on the values \( G(j\omega) \) of a transfer function
\[
G(s) = C(sI - A)^{-1}B
\]
\(^1\)Posted October 1, 2003. Due date October 8, 2003
with a Hurwitz matrix $A$, which guarantee that $x(t) \to 0$ as $t \to \infty$ for every solution of

$$
\dot{x}(t) = Ax(t) + Bw(t), \; v(t) = Cx(t), \; w(\cdot) = \Delta(v(\cdot)).
$$

Make sure that your conditions are satisfied at least for one non-zero transfer function $G = G(s)$.

**Problem 4.3**

For the pendulum equation

$$
\ddot{y}(t) + \dot{y} + \sin(y) = 0,
$$

find a single continuously differentiable Lyapunov function $V = V(y, \dot{y})$ that yields the maximal region of attraction of the equilibrium $y = \dot{y} = 0$. (In other words, the level set

$$
\{ \bar{x} \in \mathbb{R}^2 : V(\bar{x}) < 1 \}
$$

shouled be a union of disjoint open sets, one of which is the attractor $\Omega$ of the zero equilibrium, and $V(y(t), \dot{y}(t))$ shoould have negative derivative at all points of $\Omega$ except the origin.)