Problem Set 5

Problem 5.1

\(y(t) \equiv a\) is an equilibrium solution of the differential equation

\[y^{(3)}(t) + \ddot{y}(t) + \dot{y}(t) + 2\sin(y(t)) = 2\sin(a),\]

where \(a \in \mathbb{R}\) and \(y^{(3)}\) denotes the third derivative of \(y\). For which values of \(a \in \mathbb{R}\) is this equilibrium locally exponentially stable?

Problem 5.2

In order to solve a quadratic matrix equation \(X^2 + AX + B = 0\), where \(A, B\) are given \(n\)-by-\(n\) matrices and \(X\) is an \(n\)-by-\(n\) matrix to be found, it is proposed to use an iterative scheme

\[X_{k+1} = X_k^2 + AX_k + X_k + B.\]

Assume that matrix \(X_*\) satisfies \(X_*^2 + AX_* + B = 0\). What should be required of the eigenvalues of \(X_*\) and \(A + X_*\) in order to guarantee that \(X_k \to X_*\) exponentially as \(k \to \infty\) when \(\|X_0 - X_*\|\) is small enough? You are allowed to use the fact that matrix equation

\[ay + yb = 0,\]

where \(a, b, y\) are \(n\)-by-\(n\) matrices, has a non-zero solution \(y\) if and only if \(\det(sI - a) = \det(sI + b)\) for some \(s \in \mathbb{C}\).

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Problem 5.3

Use the Center manifold theory to prove local asymptotic stability of the equilibrium at the origin of the Lorentz system

\[
\begin{align*}
\dot{x} &= -\beta x + yz, \\
\dot{y} &= -\sigma y + \sigma z, \\
\dot{z} &= -yx + \rho y - z,
\end{align*}
\]

where \(\beta,\sigma\) are positive parameters and \(\rho = 1\). Estimate the rate of convergence of \(x(t), y(t), z(t)\) to zero.

Problem 5.4

Check local asymptotic stability of the periodic trajectory \(y(t) = \sin(t)\) of system

\[
\ddot{y} + \dot{y} + y^3 = -\sin(t) + \cos(t) + \sin^3(t).
\]

Problem 5.5

Find all values of parameter \(a \in \mathbb{R}\) such that every solution \(x : [0, \infty) \to \mathbb{R}^2\) of the ODE

\[
\dot{x}(t) = \epsilon \begin{bmatrix} \cos(2t) & a \\ \cos^4(t) & \sin^4(t) \end{bmatrix} x(t)
\]

converges to zero as \(t \to \infty\) when \(\epsilon > 0\) is a sufficiently small constant.