Problem Set 8\footnote{Posted November 19, 2003. Due date November 26, 2003}

Problem 8.1

Autonomous system equations have the form

\[ \ddot{y}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}' Q \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}, \]  

(8.1)

where \( y \) is the scalar output, and \( Q = Q' \) is a given symmetric 2-by-2 matrix with real coefficients.

(a) Find all \( Q \) for which there exists a \( C^\infty \) bijection \( \psi : \mathbb{R}^2 \mapsto \mathbb{R}^2 \), matrices \( A, C, \) and a \( C^\infty \) function \( \phi : \mathbb{R} \mapsto \mathbb{R}^2 \) such that \( z = \psi(y, \dot{y}) \) satisfies the ODE

\[ \dot{z}(t) = Az(t) + \phi(y(t)), \quad y(t) = Cz(t) \]

whenever \( y(\cdot) \) satisfies (8.1).

(b) For those \( Q \) found in (a), construct \( C^\infty \) functions \( F = F_Q : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}^2 \) and \( H = H_Q : \mathbb{R}^2 \mapsto \mathbb{R} \) such that \( H_Q(\eta(t)) - \dot{y}(t) \to 0 \) as \( t \to \infty \) whenever \( y : [0, \infty) \mapsto \mathbb{R} \) is a solution of (8.1), and

\[ \dot{\eta}(t) = F_Q(\eta(t), y(t)). \]
Problem 8.2

A linear control system
\[
\begin{cases}
\dot{x}_1(t) &= x_2(t) + w_1(t), \\
\dot{x}_2(t) &= -x_1(t) - x_2(t) + u + w_2(t)
\end{cases}
\]
is equipped with the nonlinear sensor
\[ y(t) = x_1(t) + \sin(x_2(t)) + w_3(t), \]
where \( w_i(\cdot) \) represent plant disturbances and sensor noise satisfying a uniform bound \( |w_i(t)| \leq d \). Design an observer of the form
\[ \dot{\eta}(t) = F(\eta(t), y(t), u(t)) \]
and constants \( d_0 > 0 \) and \( C > 0 \) such that
\[ |\eta(t) - x(t)| \leq Cd \quad \forall \ t \geq 0 \]
whenever \( \eta(0) = x(0) \) and \( d < d_0 \). (Try to make \( d_0 \) as large as possible, and \( C \) as small as possible.)

Problem 8.3

Is it true or false that the set \( \Omega = \Omega_F = \{ P \} \) of positive definite quadratic forms \( V_P (\bar{x}) = \bar{x}'P\bar{x} \), where \( P = P' > 0 \), which are valid control Lyapunov function for a given ODE model
\[ \dot{x}(t) = F(x(t), u(t)), \]
in the sense that
\[ \inf_{\bar{u} \in \mathbb{R}} \bar{x}'P F(\bar{x}, \bar{u}) \leq -|\bar{x}|^2 \quad \forall \bar{x} \in \mathbb{R}^n, \]
is linearly connected for all continuously differentiable functions \( F : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n \)? (Remember that a set \( \Omega \) of matrices is called linearly connected if for every two matrices \( P_0, P_1 \in \Omega \) there exists a continuous function \( p : [0, 1] \mapsto \Omega \) such that \( p(0) = P_0 \) and \( p(1) = P_1 \). In particular, the empty set is linearly connected.)