Problem Set 10 (due May 12, 2004) \(^1\)

Problem 10.1

For a cone \(\Delta = \{\Delta\} \) of complex \(n\)-by-\(m\) matrices, and for a complex \(m\)-by-\(n\) matrix \(M\), the quantity \(\mu_\Delta(M)\) is defined by

\[
\mu_\Delta(M) = (\inf\{\|\Delta\| : \Delta \in \Delta, \det(I - M\Delta) = 0\})^{-1}
\]

(in particular, \(\mu_\Delta(M) = 0\) is \(I - M\Delta\) invertible for all \(\Delta \in \Delta\)). Such quantity, called structured singular value of \(M\) (where \(\Delta\) is what defines the “structure”), plays an important role in analysing robust stability.

When \(\Delta\) is the cone of all matrices, \(\mu_\Delta(M)\) equals the usual largest singular number of \(M\). When \(\Delta\) is the set of all diagonal matrices with complex entries, \(\mu_\Delta(M) = \mu_C(M)\) is called the complex structured singular value. When \(\Delta\) is the set of all diagonal matrices with real entries, \(\mu_\Delta(M) = \mu_R(M)\) is called the real structured singular value.

Let \(\Delta\) be the cone of diagonal matrices with complex entries \(z_i\) such that \(\text{Re}(z_i) \geq |\text{Im}(z_i)|\). Our objective is to produce a method for estimating \(\mu_\Delta(M)\), based on semidefinite programming.

(a) Describe the set of all quadratic constraints which are satisfied for the relation between two complex numbers \(w\) and \(v\) satisfying \(w = zv\), where \(\text{Re}(z) \geq |\text{Im}(z)|\).

(b) Use the result of (a) to develop an LMI optimization algorithm for calculating an upper bound \(\hat{\mu}_\Delta(M)\) of \(\mu_\Delta(M)\) for an arbitrary \(n\)-by-\(n\) complex matrix \(M\).

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\(^1\)Version of May 6, 2004
(c) Test the upper bound on a set of randomly generated 3-by-3 and 10-by-10 complex matrices $M$. Compare $\hat{\mu}_\Delta(M)$ with $\mu_C(M)$, which can be estimated using MATLAB’s `bounds=mu(M)` (the two components of output `bounds` will be an upper and a lower bound of $\mu_C(M)$).

**Problem 10.2**

In the design setup shown on Figure 10.1, $r$ is the reference signal, $y$ is measured plant output, $u$ is control action, and $e = y - r$ is tracking error. Transfer functions $W_1$ (reference signal shaping filter), $P_0$ (nominal plant model), and $W_2$ (uncertainty weight) are given:

$$W_1(s) = \frac{1}{1 + 20s}, \quad W_2(s) = \frac{r}{s + 10}, \quad P_0(s) = \frac{s - 2}{s^2 - 1},$$

where $r > 0$ is a parameter. $\Delta = \Delta(s)$ is the normalized uncertainty, ranging over the set of all stable transfer functions with $\|\Delta\|_\infty \leq 1$. The objective is to design an LTI controller $K = K(s)$ of order not larger than 8, which stabilizes the feedback system for all possible $\Delta$ ("robust stabilization"), while trying to make the worst (again, over all possible $\Delta$) closed loop H-Infinity norm from $w$ to $e$ ("robust performance") as small as possible.

(a) Find the maximal value $r_0$ of those $r > 0$ for which robust stabilization is possible.
(b) For $r = 0.1 r_0$, use D-K iterations of H-Infinity optimization and semidefinite programming to minimize robust performance.