Problem Set 2 Solutions

Problem 2.1

For each of the statements below decide whether it is true or false. For a true statement, sketch a proof. For a false statement, give a counterexample.

(a) H2 norm of a stable finite order CT LTI state space model is never larger than 100 times its H-Infinity norm.

This statement is false. To see this, consider the stable first order LTI CT system with transfer function

\[ G(s) = G_a(s) = \frac{1}{s + a}, \]

where \( a > 0 \) is a real parameter. Then

\[ \|G_a\|_\infty = \sup_{\omega \in \mathbb{R}} |G(j\omega)| = \sup_{\omega \in \mathbb{R}} \frac{1}{\sqrt{a^2 + \omega^2}} = \frac{1}{a}, \]

\[ \|G_a\|_{H2} = \left( \int_0^\infty e^{-2at} dt \right)^{1/2} = \frac{1}{\sqrt{2a}} \]

(we used the fact that \( g_a(t) = e^{-at} \), where \( t \geq 0 \), is the impulse response of \( G_a \)). Hence, as \( a \rightarrow 0 \), H-Infinity norm of \( G_a \) becomes arbitrarily large relative the H2 norm of \( G_a \).

\[ ^1\text{Version of February 22, 2004} \]
(b) **H-Infinity norm of a stable finite order CT LTI state space model is never larger than 100 times its H2 norm.**

This statement is **false**. To see this, use $G_a$ from (a) with $a \to \infty$.

(c) **H2 norm of a stable finite order DT LTI state space model is never larger than 100 times its H-Infinity norm.**

The statement is **true** for SISO models, because

$$
\|G\|_{H2}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \max_\omega \{|G(e^{j\omega})|^2\} d\omega = \|G\|_{\infty}^2.
$$

However, for MIMO systems, the statement is **false**. To see this, consider $G = G(z)$ which is an $n$-by-$n$ identity matrix: its H-Infinity norm equals 1 while the H2 norm equals $\sqrt{n}$.

(d) **H-Infinity norm of a stable finite order DT LTI state space model is never larger than 100 times its H2 norm.**

This statement is **false**. To see this, consider the stable first order LTI DT system with transfer function

$$
H(z) = H_a(z) = \frac{1}{1 - a/z},
$$

where $a \in (0, 1)$ is a real parameter. Then

$$
\|H_a\|_{\infty} = \sup_{\omega \in \mathbb{R}} |H(e^{j\omega})| = \frac{1}{1 - a},
$$

$$
\|H_a\|_{H2} = \left( \sum_{k=0}^{\infty} a^{2k} \right)^{1/2} = \frac{1}{\sqrt{1 - a^2}}
$$

(we used the fact that $h_a[k] = a^k$, where $k \geq 0$, is the impulse response of $H_a$). Hence, as $a \to 1$, H-Infinity norm of $H_a$ becomes arbitrarily large relative the H2 norm of $H_a$.

Problem 2.2

**For continuous time (non-LTI) systems** $S_a$ **with scalar input** $f = f(t)$ **and scalar output** $g = g(t)$, **described below**, **find their L2 gains**, **as functions of parameter** $a > 0$. **Support your answer with arguments** (typically,
to show that L2 gain of a system equals, say, 0.5, one has to find input/output pairs of infinite energy (i.e., not converging to zero) for which the asymptotic (as time converges to infinity) output-to-input energy ratio is arbitrarily close to 0.25 = 0.5^2, and, in addition, to show that the asymptotic energy ratio cannot be larger than 0.5).

(a) \( g(t) = a \sin(f(t)) \);

L2 gain equals \( a \).

To see that the L2 gain cannot be larger, note that

\[ |\sin(y)| \leq |y| \]

for all real \( y \). Hence

\[ \int_{0}^{T} |g(t)|^2 dt = \int_{0}^{T} |a \sin(f(t))|^2 dt \leq a^2 \int_{0}^{T} |f(t)|^2 dt. \]

To see that the L2 gain cannot be smaller, consider input \( f(t) \equiv \epsilon \), where \( \epsilon > 0 \) is a small parameter. Then, for every \( \gamma \) we have

\[ \int_{0}^{T} \{\gamma^2 |f(t)|^2 - |g(t)|^2 \} dt = T(\gamma^2 \epsilon^2 - a^2 \sin^2(\epsilon)), \]

which will converge to minus infinity as \( T \to \infty \) unless

\[ \gamma^2 \epsilon^2 - a^2 \sin^2(\epsilon) \geq 0. \]

Since this inequality must be satisfied for all \( \epsilon \) whenever \( \gamma \) is larger than the L2 gain, we have \( \gamma^2 \geq a^2 \).

(b) \( g(t) = f(at) \sin(t) \);

For \( a \leq 1 \) L2 gain equals \( 1/\sqrt{a} \). For \( a > 1 \) the gain is infinite (and the system is not causal).

To see that L2 gain does not exceed \( 1/\sqrt{a} \) for \( a \leq 1 \), note that

\[ \int_{0}^{T} |g(t)|^2 dt = \int_{0}^{T} |f(at)|^2 \sin^2(t) dt \leq \int_{0}^{T} |f(at)|^2 dt = \frac{1}{a} \int_{0}^{aT} |f(t)|^2 dt \leq \frac{1}{a} \int_{0}^{T} |f(t)|^2 dt. \]
To see that $L^2$ gain is not smaller than $1/\sqrt{a}$ for $a \leq 1$, consider
\[
  f(t) = \begin{cases} 
    k^{-1/2}, & t \in [(k + \frac{1}{2})\pi - \epsilon, (k + \frac{1}{2})\pi + \epsilon], \ k \in \{1, 2, \ldots\}, \\
    0, & \text{otherwise}. 
  \end{cases}
\]
(The main idea is that $f(t)$ should be zero when $|\sin(t)|$ is not close to 1, and the energy of $f(t)$ on the interval $[aT, T]$ should converge to zero as $T \to \infty$, while the total energy of $f$ should be infinite.) Then
\[
  \int_0^T \{ \gamma^2 |f(t)|^2 - |g(t)|^2 \} dt \leq \int_0^T \gamma^2 |f(t)|^2 dt - \int_0^T \cos^2(\epsilon) |f(at)|^2 dt
\]
\[
  = \left( \gamma^2 - \frac{\cos^2(\epsilon)}{a} \right) \int_0^T |f(t)|^2 dt + \gamma^2 \int_0^T |f(t)|^2 dt,
\]
which converges to $-\infty$ as $T \to \infty$ unless $\gamma^2 \geq \cos^2(\epsilon)/a$. Since $\epsilon > 0$ can be arbitrarily small, $g \geq 1/a$ for $a \geq 1$.

To show that $L^2$ gain is infinite for $a > 1$, consider
\[
  f(t) = \begin{cases} 
    r^k, & t \in [(k + \frac{1}{2})\pi - \epsilon, (k + \frac{1}{2})\pi + \epsilon], \ k \in \{1, 2, \ldots\}, \\
    0, & \text{otherwise},
  \end{cases}
\]
where $r \gg 1$ is a parameter. It is easy to see that, when $r \to \infty$ is sufficiently large compared to $\gamma$, the integrals
\[
  \int_0^T \{ \gamma^2 |f(t)|^2 - |g(t)|^2 \} dt
\]
converge to minus infinity as $T \to \infty$.

(c) $g(t) = af(t) - |f(t - 1)|$.

L2 gain equals $1 + a$.

To see that $L^2$ gain does not exceed $1 + a$, note that
\[
  |ax + y|^2 \leq (1 + a) (a|x|^2 + |y|^2) \quad \forall \ x, y,
\]
and hence
\[
  \int_0^T |g(t)|^2 dt \leq (1 + a) a \int_0^T |f(t)|^2 dt + (1 + a) \int_0^T |f(t - 1)|^2 dt
\]
\[
  = (1 + a)^2 \int_0^T |f(t)|^2 dt + \int_{-1}^0 |f(t)|^2 dt.
\]
To see that $L^2$ gain is not smaller than $1 + a$, consider the case when $f(t) \equiv -1$, and hence $g(t) \equiv -(1 + a)$. 
Problem 2.3

Continuous time signal $q = q(t)$ is the output of a pure double integrator system with input $f_1 = f_1(t)$, and $g(t) = q(t) + bf_2(t)$, where $b > 0$ is a known constant (do the calculations for $b = 0.1$ and $b = 10$). Find an LTI filter $F = F(s)$ which takes $g = g(t)$ as an input and outputs an estimate $\hat{q} = \hat{q}(t)$ of $q = q(t)$, which is “good” in one of the following interpretations:

(a) Assuming that $f = [f_1; f_2]$ is white noise, minimize the asymptotic value of the variance of the estimation error $e = q - \hat{q}$.

(b) Minimize the $L_2$ gain from $f = [f_1; f_2]$ to the estimation error $e = q - \hat{q}$ with accuracy 10 percent.

For a system with state space equations

$$\dot{x}(t) = A^0 x(t) + B^0_1 w(t)$$

and sensor output

$$y_0(t) = C^0_2 x(t) + D^0_{21} w(t),$$

a standard observer has the format

$$\dot{\hat{x}}(t) = A^0 \hat{x}(t) + L(C^0_2 \hat{x}(t) - y_0(t)),$$

where matrix $L$ is chosen in such way that $A^0 + LC^0_2$ is a Hurwitz matrix. Note that here $\hat{x}$ is a result of applying an LTI transformation to $y$. Hence

$$d(t) = y_0(t) - C^0_2 \hat{x}(t)$$

is a result of applying an LTI transformation to $y_0(t)$, and, reciprocally, $y_0(t)$ is a result of applying an LTI transformation to $d(t)$. Therefore, designing an LTI filter $F$ with input $y_0$ output $v = F y_0$, to be an optimal estimate of a state component $q(t) = C^0_1 x(t)$ can be reduced to designing an LTI filter $F_d$ with input $d(t)$ and output $v_d = F_d d$, to be a good estimate of $q_d(t) = q(t) - C^0_1 \hat{x}(t)$: the relation between $v_d$ and $v$ will be

$$v(t) = C^0_1 \hat{x}(t) + v_d(t).$$

The relation between $w$, $d$, and $q_d$ is given by

$$\dot{e} = (A^0 + LC^0_2) e + (B^0_1 + LD^0_{21}) w, \quad d = C^0_2 e + D^0_{21} w, \quad q_d = C^0_1 e,$$
where $e(t) = x(t) - \hat{x}(t)$ plays the role of system state. The task of finding an optimal or suboptimal estimator for $q_d$ based on measuring $d$ can be formulated as a standard LTI feedback optimization setup with

$$A = A^0 + LC_2^0, B_1 = B_1^0 + LD_{21}^0, B_2 = 0, C_1 = C_1^0, D_11 = 0, D_{12} = I, C_2 = C_2^0, D_{21} = D_{21}^0, D_{22} = 0.$$

To implement the filter optimization approach using MATLAB, we use a design SIMULINK model ps2.3a.mdl,

handled by M-function ps2_3.md1:

```matlab
function [Fh2,Fhi,Eh2,Ehi]=ps2_3(b)
% function ps2_3(b)
% % solution for Problem 2.3 in 6.245/Spring 2004

if nargin<1, b=1; end
assignin('base','b',b);
s=tf('s');
assignin('base','s',s);
load_system('ps2_3a');
[a,b,c,d]=linmod('ps2_3a');
close_system('ps2_3a');
[ar,br,cr,dr]=ssdata(minreal(ss(a,b,c,d)));
p=pck(ar,br,cr,dr);
mmeas=1;
ncon=1;
```
ricmethd=2;
quiet=0;
[kh2,gh2]=h2syn(p,nmeas,ncon,ricmethd,quiet);
[ah2,bh2,ch2,dh2]=unpck(kh2);
[ag,bg,cg,dg]=unpck(gh2);
Kh2=ss(ah2,bh2,ch2,dh2);
G=(s+1)/(s^2+s+1);
disp('H2 controller:');
Fh2=tf(minreal(Kh2*(G-1)+G))
Eh2=tf(minreal(ss(ag,bg,cg,dg)));
gmin=0;
gmax=norm(Eh2,Inf);
tol=0.01;
epr=1e-10;
epp=1e-6;
[khinf,ghinf]=hinfsyn(p,nmeas,ncon,gmin,gmax,tol,ricmethd,epr,epp,quiet);
[ahi,bhi,chi,dhi]=unpck(khinf);
[ag,bg,cg,dg]=unpck(ghinf);
Ehi=tf(minreal(ss(ag,bg,cg,dg)));
Khi=ss(ahi,bhi,chi,dhi);
disp('H-Infinity controller:');
Fhi=tf(minreal(Khi*(G-1)+G))

Note the need for using minreal.m: MATLAB does not eliminate uncontrollable/unobservable states automatically.