Problem Set 4 (due March 3, 2004) \(^1\)

**Problem 4.1**

For the SISO feedback design from Figure 4.1, where it is known that \(P(2) = 0\) and \(1 \pm 2j\) are poles of \(P\), find a lower bound (as good as you can) on the H-Infinity norm of the closed-loop complementary sensitivity transfer function \(T = T(s)\) (from \(r\) to \(v\)), assuming that \(C = C(s)\) is a stabilizing controller, \(|T(j\omega) - 1| < 0.2\) for \(|\omega| < 10\), and \(|T(j\omega)| < 0.1\) for \(|\omega| > 20\).

![Figure 4.1: A SISO Feedback Setup](image)

**Problem 4.2**

(a) Apply the formulae for H2 optimization to a standard setup with a Hurwitz matrix \(A\) and with \(B_2 = 0\) to express the H2 norm of CT LTI MIMO state space model

\[
y = Cx, \quad \dot{x} = Ax + Bf
\]

\(^1\)Version of February 25, 2004
in terms of matrices $C$ and $P$, where $P = P'$ is the solution of the Lyapunov equation
\[ AP + PA' = -BB'. \]

(b) Use the result from (a) to obtain an explicit (with respect to $a_0, a_1, a_2$) formula for the H2 norm of system with transfer function
\[ G(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}, \]
where $a_0, a_1, a_2$ are positive real numbers such that $a_1 a_2 > a_0$.

(c) Use MATLAB to check numerically correctness of your analytical solution.

**Problem 4.3**

An undamped linear oscillator with 3 degrees of freedom is described as a SISO LTI system with control input $v$, “position” output $q$, and transfer function
\[ P_0(s) = \frac{1}{(1 + s^2)(4 + s^2)(16 + s^2)}. \]
The position output $q$ is measured with delay and noise, so that the sensor output $g$ is defined as $g = 0.1 f_1 + \hat{q}$, where $\hat{q}$ is output of an LTI system with input $q$ and transfer function
\[ P_1(s) = \frac{1 - s}{1 + s}. \]
The task is to design an LTI controller which takes $g$ and a scalar reference signal $r$ as inputs, produces $v$ as its output, and satisfies the following specifications, assuming $r$ is the output of an LTI system with input $f_2$ and transfer function
\[ P_2(s) = \frac{\sqrt{2a}}{s + a}, \]
where $a > 0$ is a parameter, and $f = [f_1; f_2]$ is a normalized white noise:

(a) the closed loop system is stable;

(b) the closed loop dc gain from $r$ to $q$ equals 1;

(c) the mean square value of control signal $v$ is within 100;

(d) the mean square value of tracking error $e = q - r$ is as small as possible (within 20 percent of its minimum subject to constraints (a)-(c)).

Use H2 optimization to solve the problem for $a = 0.2$ and $a = 0.01$. 