15.081J/6.251J Introduction to Mathematical Programming

Lecture 9: Duality Theory II
1 Outline
- Strict complementary slackness
- Geometry of duality
- The dual simplex algorithm
- Duality and degeneracy

2 Strict Complementary Slackness
Assume that both problems have an optimal solution:
\[
\begin{align*}
\min & \quad c'x \\
\text{s.t.} & \quad Ax \geq b \\
\end{align*}
\quad \begin{align*}
\max & \quad p'b \\
\text{s.t.} & \quad p'A \leq c' \\
& \quad p \geq 0.
\end{align*}
\]
There exist optimal solutions to the primal and to the dual that satisfy
- For every \( j \), either \( x_j > 0 \) or \( p'A_j < c_j \).
- For every \( i \), we have either \( a'_i x > b_i \) or \( p_i > 0 \).

2.1 Example
\[
\begin{align*}
\min & \quad 5x_1 + 5x_2 \\
\text{s.t.} & \quad x_1 + x_2 \geq 2 \\
& \quad 2x_1 - x_2 \geq 0 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]
- Is \((2/3, 4/3)\) strictly complementary?
- Which are all the strictly complementary solutions?

3 The Geometry of Duality
\[
\begin{align*}
\min & \quad c'x \\
\text{s.t.} & \quad a'_i x \geq b_i, \quad i = 1, \ldots, m \\
\max & \quad p'b \\
\text{s.t.} & \quad \sum_{i=1}^m p_i a_i = c \\
& \quad p \geq 0
\end{align*}
\]
4 Dual Simplex Algorithm

4.1 Motivation

- In simplex method $B^{-1}b \geq 0$
- Primal optimality condition $c' - c'_B B^{-1}A \geq 0'$

same as dual feasibility
- Simplex is a primal algorithm: maintains primal feasibility and works towards dual feasibility
- Dual algorithm: maintains dual feasibility and works towards primal feasibility

\[
\begin{array}{|c|c|c|}
\hline
-x_B' x_B & c_1 & \ldots & c_m \\
\hline
x_{B(1)} & | & | \\
\vdots & | & | \\
x_{B(m)} & | & | \\
\hline
\end{array}
\]

\[B^{-1}A_1 \ldots B^{-1}A_m\]

- Do not require $B^{-1}b \geq 0$
- Require $\bar{c} \geq 0$ (dual feasibility)
- Dual cost is $p'b = c'_B B^{-1}b = c'_B x_B$
- If $B^{-1}b \geq 0$ then both dual feasibility and primal feasibility, and also same cost $\Rightarrow$ optimality
- Otherwise, change basis

4.2 An iteration

1. Start with basis matrix $B$ and all reduced costs $\geq 0$.

2. If $B^{-1}b \geq 0$ optimal solution found; else, choose $l$ s.t. $x_{B(l)} < 0$.

3. Consider the $l$th row (pivot row) $x_{B(l)}, v_1, \ldots, v_n$. If $\forall i v_i \geq 0$ then dual optimal cost = $+\infty$ and algorithm terminates.

4. Else, let $j$ s.t.

\[
\frac{\bar{c}_j}{|v_j|} = \min_{\{i | v_i < 0\}} \frac{\bar{c}_i}{|v_i|}
\]

5. Pivot element $v_j$: $A_j$ enters the basis and $A_{B(l)}$ exits.
4.3 An example

\[
\begin{align*}
\text{min} & \quad x_1 + x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 \geq 2 \\
& \quad x_1 \geq 1 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad x_1 + x_2 & \quad \text{max} & \quad 2p_1 + p_2 \\
\text{s.t.} & \quad x_1 + 2x_2 - x_3 = 2 & \text{s.t.} & \quad p_1 + p_2 \leq 1 \\
& \quad x_1 - x_4 = 1 & \quad & \quad 2p_1 \leq 1 \\
& \quad x_1, x_2, x_3, x_4 \geq 0 & \quad & \quad p_1, p_2 \geq 0
\end{align*}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & x_1 & x_2 & x_3 & x_4 \\
\hline
x_3 = & -2 & -1 & -2^* & 1 & 0 \\
x_4 = & -1 & -1 & 0 & 0 & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & x_1 & x_2 & x_3 & x_4 \\
\hline
x_2 = & -1 & 1/2 & 0 & 1/2 & 0 \\
x_4 = & 1 & 1/2 & 1 & -1/2 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & x_1 & x_2 & x_3 & x_4 \\
\hline
x_2 = & -3/2 & 0 & 0 & 1/2 & 1/2 \\
x_1 = & 1/2 & 0 & 1 & -1/2 & 1/2 \\
\hline
\end{array}
\]

4
5 Duality and Degeneracy

- Any basis matrix $B$ leads to dual basic solution $p' = c_B'B^{-1}$.

- The dual constraint $p'A_j = c_j$ is active if and only if the reduced cost $\bar{r}_j$ is zero.

- Since $p$ is $m$-dimensional, dual degeneracy implies more than $m$ reduced costs that are zero.

- Dual degeneracy is obtained whenever there exists a nonbasic variable whose reduced cost is zero.

5.1 Example

$$\begin{align*}
\text{min} & \quad 3x_1 + x_2 \quad \text{max} & \quad 2p_1 \\
\text{s.t.} & \quad x_1 + x_2 - x_3 = 2 \quad \text{s.t.} & \quad p_1 + 2p_2 \leq 3 \\
& \quad 2x_1 - x_2 - x_4 = 0 \quad & \quad p_1 - p_2 \leq 1 \\
& \quad x_1, x_2, x_3, x_4 \geq 0, \quad & \quad p_1, p_2 \geq 0.
\end{align*}$$

Equivalent primal problem

$$\begin{align*}
\text{min} & \quad 3x_1 + x_2 \\
\text{s.t.} & \quad x_1 + x_2 \geq 2 \\
& \quad 2x_1 - x_2 \geq 0 \\
& \quad x_1, x_2 \geq 0.
\end{align*}$$

- Four basic solutions in primal: $A, B, C, D$.


- Different bases may lead to the same basic solution for the primal, but to different basic solutions for the dual. Some are feasible and some are infeasible.
5.2 Degeneracy and uniqueness

- If dual has a nondegenerate optimal solution, the primal problem has a unique optimal solution.

- It is possible, however, that dual has a degenerate solution and the dual has a unique optimal solution.