

6.252 NONLINEAR PROGRAMMING

LECTURE 14: INTRODUCTION TO DUALITY

LECTURE OUTLINE

• Convex Cost/Linear Constraints
• Duality Theorem
• Linear Programming Duality
• Quadratic Programming Duality

Linear inequality constrained problem

\[ \text{minimize } f(x) \]
\[ \text{subject to } a_j'x \leq b_j, \quad j = 1, \ldots, r, \]

where \( f \) is convex and continuously differentiable over \( \mathbb{R}^n \).
LAGRANGE MULTIPLIER RESULT

Let $J \subset \{1, \ldots, r\}$. Then $x^*$ is a global min if and only if $x^*$ is feasible and there exist $\mu_j^* \geq 0$, $j \in J$, such that $\mu_j^* = 0$ for all $j \in J \notin A(x^*)$, and

$$x^* = \arg \min_{\substack{a_j'x \leq b_j \\ j \notin J}} \left\{ f(x) + \sum_{j \in J} \mu_j^*(a_j'x - b_j) \right\}.$$

Proof: Assume $x^*$ is global min. Then there exist $\mu_j^* \geq 0$, such that $\mu_j^*(a_j'x^* - b_j) = 0$ for all $j$ and

$$\nabla f(x^*) + \sum_{j=1}^{r} \mu_j^* a_j = 0,$$

implying

$$x^* = \arg \min_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^*(a_j'x - b_j) \right\}.$$

Since $\mu_j^*(a_j'x^* - b_j) = 0$ for all $j$,

$$f(x^*) = \min_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^*(a_j'x - b_j) \right\}.$$

Since $\mu_j^*(a_j'x - b_j) \leq 0$ if $a_j'x - b_j \leq 0$,

$$f(x^*) \leq \min_{\substack{a_j'x \leq b_j \\ j \notin J}} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^*(a_j'x - b_j) \right\} \leq \min_{\substack{a_j'x \leq b_j \\ j \notin J}} \left\{ f(x) + \sum_{j \in J} \mu_j^*(a_j'x - b_j) \right\}.$$
PROOF (CONTINUED)

Conversely, if \( x^* \) is feasible and there exist scalars \( \mu_j^* \), \( j \in J \) with the stated properties, then

\[
\left( \nabla f(x^*) + \sum_{j \in J} \mu_j^* a_j \right)' (x - x^*) \geq 0, \quad \text{if } a_j' x \leq b_j, \ \forall j \notin J.
\]

For all \( x \) that are feasible for the original problem, \( a_j' x \leq b_j = a_j' x^* \) for all \( j \in A(x^*) \). Since \( \mu_j^* = 0 \) if \( j \in J \) and \( j \notin A(x^*) \),

\[
\sum_{j \in J} \mu_j^* a_j' (x - x^*) \leq 0,
\]

which implies

\[
\nabla f(x^*)' (x - x^*) \geq 0
\]

for all feasible \( x \). Hence \( x^* \) is a global min. \ Q.E.D. \n
- Note that the same set of \( \mu_j^* \) works for all index sets \( J \).
THE DUAL PROBLEM

• Consider the problem

$$\min_{x \in X, a'_j x \leq b_j, \ j=1,\ldots,r} f(x)$$

where $f$ is convex and cont. differentiable over $\mathbb{R}^n$ and $X$ is polyhedral.

• Define the dual function $q : \mathbb{R}^r \mapsto [\infty, \infty)$

$$q(\mu) = \inf_{x \in X} L(x, \mu) = \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^{r} \mu_j (a'_j x - b_j) \right\}$$

and the dual problem

$$\max_{\mu \geq 0} q(\mu).$$

• If $X$ is bounded, the dual function takes real values. In general, $q(\mu)$ can take the value $-\infty$. The “effective” constraint set of the dual is $Q = \{ \mu \mid \mu \geq 0, \ q(\mu) > -\infty \}$. 
DUALITY THEOREM

(a) If the primal problem has an optimal solution, the dual problem also has an optimal solution and the optimal values are equal.
(b) $x^*$ is primal-optimal and $\mu^*$ is dual-optimal if and only if $x^*$ is primal-feasible, $\mu^* \geq 0$, and

$$f(x^*) = L(x^*, \mu^*) = \min_{x \in X} L(x, \mu^*).$$

Proof: (a) Let $x^*$ be a primal optimal solution. For all primal feasible $x$, and all $\mu \geq 0$, we have $\mu_j'(a_j'x - b_j) \leq 0$ for all $j$, so

$$q(\mu) \leq \inf_{x \in X, a_j'x \leq b_j} \left\{ f(x) + \sum_{j=1}^{r} \mu_j(a_j'x - b_j) \right\} \leq \inf_{x \in X, a_j'x \leq b_j} f(x) = f(x^*).$$

By L-Mult. Th., there exists $\mu^* \geq 0$ such that $\mu_j^*(a_j'x^* - b_j) = 0$ for all $j$, and $x^* = \arg \min_{x \in X} L(x, \mu^*)$, so

$$q(\mu^*) = L(x^*, \mu^*) = f(x^*) + \sum_{j=1}^{r} \mu_j^*(a_j'x^* - b_j) = f(x^*).$$
PROOF (CONTINUED)

(b) If \( x^* \) is primal-optimal and \( \mu^* \) is dual-optimal, by part (a)

\[
f(x^*) = q(\mu^*),
\]

which when combined with Eq. (*), yields

\[
f(x^*) = L(x^*, \mu^*) = q(\mu^*) = \min_{x \in X} L(x, \mu^*).
\]

Conversely, the relation \( f(x^*) = \min_{x \in X} L(x, \mu^*) \) is written as \( f(x^*) = q(\mu^*) \), and since \( x^* \) is primal-feasible and \( \mu^* \geq 0 \), Eq. (*) implies that \( x^* \) is primal-optimal and \( \mu^* \) is dual-optimal. Q.E.D.

- Linear equality constraints are treated similar to inequality constraints, except that the sign of the Lagrange multipliers is unrestricted:

\[
\text{Primal:} \quad \min_{x \in X, e'_i x = d_i, i=1,...,m, a'_j x \leq b_j, j=1,...,r} f(x)
\]

\[
\text{Dual:} \quad \max_{\lambda \in \mathbb{R}^m, \mu \geq 0} q(\lambda, \mu) = \max_{\lambda \in \mathbb{R}^m, \mu \geq 0} \inf_{x \in X} L(x, \lambda, \mu).
\]
THE DUAL OF A LINEAR PROGRAM

• Consider the linear program

  \[ \begin{align*}
  & \text{minimize} \quad c'x \\
  & \text{subject to} \quad e_i'x = d_i, \quad i = 1, \ldots, m, \quad x \geq 0
  \end{align*} \]

• Dual function

  \[ q(\lambda) = \inf_{x \geq 0} \left\{ \sum_{j=1}^{n} \left( c_j - \sum_{i=1}^{m} \lambda_i e_{ij} \right) x_j + \sum_{i=1}^{m} \lambda_i d_i \right\}. \]

• If \( c_j - \sum_{i=1}^{m} \lambda_i e_{ij} \geq 0 \) for all \( j \), the infimum is attained for \( x = 0 \), and \( q(\lambda) = \sum_{i=1}^{m} \lambda_i d_i \). If \( c_j - \sum_{i=1}^{m} \lambda_i e_{ij} < 0 \) for some \( j \), the expression in braces can be arbitrarily small by taking \( x_j \) suff. large, so \( q(\lambda) = -\infty \). Thus, the dual is

  \[ \begin{align*}
  & \text{maximize} \quad \sum_{i=1}^{m} \lambda_i d_i \\
  & \text{subject to} \quad \sum_{i=1}^{m} \lambda_i e_{ij} \leq c_j, \quad j = 1, \ldots, n.
  \end{align*} \]
THE DUAL OF A QUADRATIC PROGRAM

• Consider the quadratic program

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x' Q x + c' x \\
\text{subject to} & \quad Ax \leq b,
\end{align*}
\]

where \( Q \) is a given \( n \times n \) positive definite symmetric matrix, \( A \) is a given \( r \times n \) matrix, and \( b \in \mathbb{R}^r \) and \( c \in \mathbb{R}^n \) are given vectors.

• Dual function:

\[
q(\mu) = \inf_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} x' Q x + c' x + \mu'(Ax - b) \right\}.
\]

The infimum is attained for \( x = -Q^{-1}(c + A'\mu) \), and, after substitution and calculation,

\[
q(\mu) = -\frac{1}{2} \mu' A Q^{-1} A' \mu - \mu'(b + A Q^{-1} c) - \frac{1}{2} c' Q^{-1} c.
\]

• The dual problem, after a sign change, is

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \mu' P \mu + t' \mu \\
\text{subject to} & \quad \mu \geq 0,
\end{align*}
\]

where \( P = A Q^{-1} A' \) and \( t = b + A Q^{-1} c \).