Strong Duality Theorem

Linear equality constraints. Fenchel Duality.

Consider the problem

\[ \begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in X, \quad g_j(x) \leq 0, \quad j = 1, \ldots, r,
\end{align*} \]

assuming \(-\infty < f^* < \infty\).

\(\mu^*\) is a Lagrange multiplier if \(\mu^* \geq 0\) and \(f^* = \inf_{x \in X} L(x, \mu^*)\).

Dual problem: Maximize \(q(\mu) = \inf_{x \in X} L(x, \mu)\) subject to \(\mu \geq 0\).
DUALITY THEOREM FOR INEQUALITIES

- Assume that $X$ is convex and the functions $f : \mathbb{R}^n \mapsto \mathbb{R}$, $g_j : \mathbb{R}^n \mapsto \mathbb{R}$ are convex over $X$. Furthermore, the optimal value $f^*$ is finite and there exists a vector $\bar{x} \in X$ such that

$$g_j(\bar{x}) < 0, \quad \forall \ j = 1, \ldots, r.$$ 

- Strong Duality Theorem: There exists at least one Lagrange multiplier and there is no duality gap.
PROOF OUTLINE

• Show that \( A \) is convex. [Consider vectors \((z, w) \in A \) and \((\tilde{z}, \tilde{w}) \in A \), and show that their convex combinations lie in \( A \).]

• Observe that \((0, f^*)\) is not an interior point of \( A \).

• Hence, there is hyperplane passing through \((0, f^*)\) and containing \( A \) in one of the two corresponding halfspaces; i.e., a \((\mu, \beta) \neq (0, 0)\) with

\[
\beta f^* \leq \beta w + \mu^' z, \quad \forall (z, w) \in A.
\]

This implies that \( \beta \geq 0 \), and \( \mu_j \geq 0 \) for all \( j \).

• Prove that hyperplane is nonvertical, i.e., \( \beta > 0 \).

• Normalize \((\beta = 1)\), take the infimum over \( x \in X \), and use the fact \( \mu \geq 0 \), to obtain

\[
f^* \leq \inf_{x \in X} \left\{ f(x) + \mu^' g(x) \right\} = q(\mu) \leq \sup_{\mu \geq 0} q(\mu) = q^*.
\]

Using the weak duality theorem, \( \mu \) is a Lagrange multiplier and there is no duality gap.
LINEAR EQUALITY CONSTRAINTS

• Suppose we have the additional constraints

\[ e_i' x - d_i = 0, \quad i = 1, \ldots, m \]

• We need the notion of the affine hull of a convex set \( X \) [denoted \( \text{aff}(X) \)]. This is the intersection of all hyperplanes containing \( X \).

• The relative interior of \( X \), denoted \( \text{ri}(X) \), is the set of all \( x \in X \) s.t. there exists \( \epsilon > 0 \) with

\[ \{ z \mid \|z - x\| < \epsilon, \ z \in \text{aff}(X) \} \subset X, \]

that is, \( \text{ri}(X) \) is the interior of \( X \) relative to \( \text{aff}(X) \).

• Every nonempty convex set has a nonempty relative interior.
DUALITY THEOREM FOR EQUALITIES

• Assumptions:
  – The set $X$ is convex and the functions $f, g_j$ are convex over $X$.
  – The optimal value $f^*$ is finite and there exists a vector $\bar{x} \in r_i(X)$ such that
    
    \[ g_j(\bar{x}) < 0, \quad j = 1, \ldots, r, \]
    
    \[ e_i' \bar{x} - d_i = 0, \quad i = 1, \ldots, m. \]

• Under the preceding assumptions there exists at least one Lagrange multiplier and there is no duality gap.
COUNTEREXAMPLE

• Consider

\[
\begin{align*}
\text{minimize} \quad & f(x) = x_1 \\
\text{subject to} \quad & x_2 = 0, \quad x \in X = \{(x_1, x_2) \mid x_1^2 \leq x_2\}.
\end{align*}
\]

• The optimal solution is \(x^* = (0, 0)\) and \(f^* = 0\).

• The dual function is given by

\[
q(\lambda) = \inf_{x_1^2 \leq x_2} \{x_1 + \lambda x_2\} = \begin{cases} 
-\frac{1}{4\lambda}, & \text{if } \lambda > 0, \\
-\infty, & \text{if } \lambda \leq 0.
\end{cases}
\]

• No dual optimal solution and therefore there is no Lagrange multiplier. (Even though there is no duality gap.)

• Assumptions are violated (the feasible set and the relative interior of \(X\) have no common point).
FENCHEL DUALITY FRAMEWORK

• Consider the problem

\[
\begin{align*}
\text{minimize } & f_1(x) - f_2(x) \\
\text{subject to } & x \in X_1 \cap X_2,
\end{align*}
\]

where \( f_1 \) and \( f_2 \) are real-valued functions on \( \mathbb{R}^n \), and \( X_1 \) and \( X_2 \) are subsets of \( \mathbb{R}^n \).

• Assume that \(-\infty < f^* < \infty\).

• Convert problem to

\[
\begin{align*}
\text{minimize } & f_1(y) - f_2(z) \\
\text{subject to } & z = y, \quad y \in X_1, \quad z \in X_2,
\end{align*}
\]

and dualize the constraint \( z = y \).

\[
q(\lambda) = \inf_{y \in X_1, z \in X_2} \left\{ f_1(y) - f_2(z) + (z - y)'\lambda \right\}
\]

\[
= \inf_{z \in X_2} \left\{ z'\lambda - f_2(z) \right\} - \sup_{y \in X_1} \left\{ y'\lambda - f_1(y) \right\}
\]

\[
= g_2(\lambda) - g_1(\lambda)
\]
DUALITY THEOREM

\[ f^* = \max_{\lambda \in \mathbb{R}^n} \left\{ g_2(\lambda) - g_1(\lambda) \right\} \]

and that the maximum above is attained.