Consider the primal problem

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in X, \quad g_j(x) \leq 0, \quad j = 1, \ldots, r,
\end{align*}
\]

assuming \(-\infty < f^* < \infty\).

Dual problem: Maximize

\[
q(\mu) = \inf_{x \in X} L(x, \mu) = \inf_{x \in X} \{f(x) + \mu' g(x)\}
\]

subject to \(\mu \in M = \{\mu \mid \mu \geq 0, \ q(\mu) > -\infty\}\).
CUTTING PLANE METHOD

- $k$th iteration, after $\mu^i$ and $g^i = g(x_{\mu^i})$ have been generated for $i = 0, \ldots, k - 1$: Solve

$$\max_{\mu \in M} Q^k(\mu)$$

where

$$Q^k(\mu) = \min_{i=0, \ldots, k-1} \left\{ q(\mu^i) + (\mu - \mu^i)'g^i \right\}.$$

Set

$$\mu^k = \arg \max_{\mu \in M} Q^k(\mu).$$
POLYHEDRAL CASE

\[ q(\mu) = \min_{i \in I} \left\{ a_i' \mu + b_i \right\} \]

where \( I \) is a finite index set, and \( a_i \in \mathbb{R}^r \) and \( b_i \) are given.

- Then subgradient \( g^k \) in the cutting plane method is a vector \( a_{i_k} \) for which the minimum is attained.
- Finite termination expected.
CONVERGENCE

• Proposition: Assume that the min of $Q_k$ over $M$ is attained and that $q$ is real-valued. Then every limit point of a sequence $\{\mu^k\}$ generated by the cutting plane method is a dual optimal solution.

Proof: $g^i$ is a subgradient of $q$ at $\mu^i$, so

$$q(\mu^i) + (\mu - \mu^i)'g^i \geq q(\mu), \quad \forall \mu \in M,$$

$$Q^k(\mu^k) \geq Q^k(\mu) \geq q(\mu), \quad \forall \mu \in M. \quad (1)$$

• Suppose $\{\mu^k\}_K$ converges to $\bar{\mu}$. Then, $\bar{\mu} \in M$, and from (1), we obtain for all $k$ and $i < k$,

$$q(\mu^i) + (\mu^k - \mu^i)'g^i \geq Q^k(\mu^k) \geq Q^k(\bar{\mu}) \geq q(\bar{\mu}).$$

• Take the limit as $i \to \infty$, $k \to \infty$, $i \in K$, $k \in K$,

$$\lim_{k \to \infty, k \in K} Q^k(\mu^k) = q(\bar{\mu}).$$

Combining with (1), $q(\bar{\mu}) = \max_{\mu \in M} q(\mu)$. 
LAGRANGIAN RELAXATION

• Solving the dual of the separable problem

\[
\text{minimize } \sum_{j=1}^{J} f_j(x_j) \\
\text{subject to } x_j \in X_j, \ j = 1, \ldots, J, \ \sum_{j=1}^{J} A_j x_j = b.
\]

• Dual function is

\[
q(\lambda) = \sum_{j=1}^{J} \min_{x_j \in X_j} \left\{ f_j(x_j) + \lambda' A_j x_j \right\} - \lambda' b \\
= \sum_{j=1}^{J} \left\{ f_j(x_j(\lambda)) + \lambda' A_j x_j(\lambda) \right\} - \lambda' b
\]

where \( x_j(\lambda) \) attains the min. A subgradient at \( \lambda \) is

\[
g_\lambda = \sum_{j=1}^{J} A_j x_j(\lambda) - b.
\]
DANTSIG-WOLFE DECOMPOSITION

- D-W decomposition method is just the cutting plane applied to the dual problem $\max_\lambda q(\lambda)$.
- At the $k$th iteration, we solve the “approximate dual”

$$\lambda^k = \arg \max_{\lambda \in \mathbb{R}^r} Q^k(\lambda) \equiv \min_{i=0, \ldots, k-1} \left\{ q(\lambda^i) + (\lambda - \lambda^i)'g^i \right\}.$$

- Equivalent linear program in $v$ and $\lambda$

maximize $v$

subject to $v \leq q(\lambda^i) + (\lambda - \lambda^i)'g^i$, $i = 0, \ldots, k - 1$

The dual of this (called master problem) is

minimize $\sum_{i=0}^{k-1} \xi^i \left( q(\lambda^i) - \lambda^i'g^i \right)$

subject to $\sum_{i=0}^{k-1} \xi^i = 1$, $\sum_{i=0}^{k-1} \xi^i g^i = 0$, $\xi^i \geq 0$, $i = 0, \ldots, k - 1$,
DANTSIG-WOLFE DECOMPOSITION (CONT.)

• The master problem is written as

\[
\text{minimize} \quad \sum_{j=1}^{J} \left( \sum_{i=0}^{k-1} \xi^i f_j(x_j(\lambda^i)) \right) \\
\text{subject to} \quad \sum_{i=0}^{k-1} \xi^i = 1, \quad \sum_{j=1}^{J} A_j \left( \sum_{i=0}^{k-1} \xi^i x_j(\lambda^i) \right) = b, \\
\xi^i \geq 0, \quad i = 0, \ldots, k - 1.
\]

• The primal cost function terms \( f_j(x_j) \) are approximated by

\[
\sum_{i=0}^{k-1} \xi^i f_j(x_j(\lambda^i))
\]

• Vectors \( x_j \) are expressed as

\[
\sum_{i=0}^{k-1} \xi^i x_j(\lambda^i)
\]
GEOMETRICAL INTERPRETATION

- Geometric interpretation of the master problem (the dual of the approximate dual solved in the cutting plane method) is *inner linearization*.

  ![Diagram](image)

- This is a “dual” operation to the one involved in the cutting plane approximation, which can be viewed as *outer linearization*.