Problem 1

(a) Let $C$ be a nonempty subset of $\mathbf{R}^n$, and let $\lambda_1$ and $\lambda_2$ be positive scalars. Show that if $C$ is convex, then $(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C$. Show by example that this need not be true when $C$ is not convex.
(b) Show that the intersection $\cap_{i \in I} C_i$ of a collection $\{C_i \mid i \in I\}$ of cones is a cone.
(c) Show that the image and the inverse image of a cone under a linear transformation is a cone.
(d) Show that the vector sum $C_1 + C_2$ of two cones $C_1$ and $C_2$ is a cone.
(e) Show that a subset $C$ is a convex cone if and only if it is closed under addition and positive scalar multiplication, i.e., $C + C \subseteq C$, and $\gamma C \subseteq C$ for all $\gamma > 0$.

Problem 2

Let $C$ be a nonempty convex subset of $\mathbf{R}^n$. Let also $f = (f_1, \ldots, f_m)$, where $f_i : C \mapsto \mathbb{R}$, $i = 1, \ldots, m$, are convex functions, and let $g : \mathbf{R}^m \mapsto \mathbb{R}$ be a function that is convex and monotonically nondecreasing over a convex set that contains the set $\{f(x) \mid x \in C\}$, in the sense that for all $u_1, u_2$ in this set such that $u_1 \leq u_2$, we have $g(u_1) \leq g(u_2)$. Show that the function $h$ defined by $h(x) = g(f(x))$ is convex over $C$. If in addition, $m = 1$, $g$ is monotonically increasing and $f$ is strictly convex, then $h$ is strictly convex.

Problem 3

Show that the following functions from $\mathbf{R}^n$ to $(-\infty, \infty]$ are convex:
(a) $f_1(x) = \ln(e^{x_1} + \cdots + e^{x_n})$.
(b) $f_2(x) = \|x\|^p$ with $p \geq 1$.
(c) $f_3(x) = e^{\beta x^TAx}$, where $A$ is a positive semidefinite symmetric $n \times n$ matrix and $\beta$ is a positive scalar.
(d) $f_4(x) = f(Ax + b)$, where $f : \mathbf{R}^m \mapsto \mathbb{R}$ is a convex function, $A$ is an $m \times n$ matrix, and $b$ is a vector in $\mathbf{R}^m$.

Problem 4

Let $X$ be a nonempty bounded subset of $\mathbf{R}^n$. Show that

$$\text{cl}(\text{conv}(X)) = \text{conv}(\text{cl}(X)).$$

In particular, if $X$ is compact, then $\text{conv}(X)$ is compact.
Problem 5

Construct an example of a point in a nonconvex set $X$ that has the prolongation property, but is not a relative interior point of $X$. 