Problem 1

(a) Show that a nonpolyhedral closed convex cone need not be retractive, by using as an example the cone $C = \{(u, v, w) \mid \|(u, v)\| \leq w\}$, the recession direction $d = (1, 0, 1)$, and the corresponding asymptotic sequence $\{(k, \sqrt{k}, \sqrt{k^2 + k})\}$. (This is the, so-called, second order cone, which plays an important role in conic programming; see Chapter 5.)

(b) Verify that the cone $C$ of part (a) can be written as the intersection of an infinite number of closed halfspaces, thereby showing that a nested set sequence obtained by intersection of an infinite number of retractive nested set sequences need not be retractive.

Solution.

(a) Clearly, $d = (1, 0, 1)$ is the recession direction associated with the asymptotic sequence $\{x_k\}$, where $x_k = (k, \sqrt{k}, \sqrt{k^2 + k})$. On the other hand, it can be verified by straightforward calculation that the vector

$$x_k - d = (k - 1, \sqrt{k}, \sqrt{k^2 + k} - 1)$$

does not belong to $C$. Indeed, denoting

$$u_k = k - 1, \quad v_k = \sqrt{k}, \quad w_k = \sqrt{k^2 + k} - 1,$$

we have

$$\|(u_k, v_k)\|^2 = (k - 1)^2 + k = k^2 - k + 1,$$

while

$$w_k^2 = (\sqrt{k^2 + k} - 1)^2 = k^2 + k + 1 - 2\sqrt{k^2 + k},$$

and it can be seen that

$$\|(u_k, v_k)\|^2 > w_k^2, \quad \forall \ k \geq 1.$$

(b) Since by the Schwarz inequality, we have

$$\max_{\|(x,y)\|=1} (ux + vy) = \|(u, v)\|,$$

it follows that the cone

$$C = \{(u, v, w) \mid \|(u, v)\| \leq w\}$$

can be written as

$$C = \cap_{\|(x,y)\|=1} \{(u, v, w) \mid ux + vy \leq w\}.$$  

Hence $C$ is the intersection of an infinite number of closed halfspaces.
Problem 2

Let $C$ be a nonempty convex set in $\mathbb{R}^n$, and let $M$ be a nonempty affine set in $\mathbb{R}^n$. Show that $M \cap \text{rin}(C) = \emptyset$ is a necessary and sufficient condition for the existence of a hyperplane $H$ containing $M$, and such that $\text{rin}(C)$ is contained in one of the open halfspaces associated with $H$.

Solution.

If there exists a hyperplane $H$ with the properties stated, the condition $M \cap \text{rin}(C) = \emptyset$ clearly holds. Conversely, if $M \cap \text{rin}(C) = \emptyset$, then $M$ and $C$ can be properly separated. This hyperplane can be chosen to contain $M$ since $M$ is affine. If this hyperplane contains a point in $\text{rin}(C)$, then it must contain all of $C$. This contradicts the proper separation property, thus showing that $\text{rin}(C)$ is contained in one of the open halfspaces.
Problem 3

Let \( C_1 \) and \( C_2 \) be nonempty convex subsets of \( \mathbb{R}^n \), and let \( B \) denote the unit ball in \( \mathbb{R}^n \), \( B = \{ x \mid \|x\| \leq 1 \} \). A hyperplane \( H \) is said to separate strongly \( C_1 \) and \( C_2 \) if there exists an \( \epsilon > 0 \) such that \( C_1 + \epsilon B \) is contained in one of the open halfspaces associated with \( H \) and \( C_2 + \epsilon B \) is contained in the other. Show that:

(a) The following three conditions are equivalent.
   (i) There exists a hyperplane separating strongly \( C_1 \) and \( C_2 \).
   (ii) There exists a vector \( \alpha \in \mathbb{R}^n \) such that \( \inf_{x \in C_1} \alpha'x > \sup_{x \in C_2} \alpha'x \).
   (iii) \( \inf_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\| > 0 \), i.e., \( 0 \notin \text{cl}(C_2 - C_1) \).

(b) If \( C_1 \) and \( C_2 \) are disjoint, any one of the five conditions for strict separation, given in Prop. 1.5.3, implies that \( C_1 \) and \( C_2 \) can be strongly separated.

Solution.
(a) We first show that (i) implies (ii). Suppose that \( C_1 \) and \( C_2 \) can be separated strongly. By definition, this implies that for some nonzero vector \( a \in \mathbb{R}^n \), \( b \in \mathbb{R} \), and \( \epsilon > 0 \), we have

\[
\begin{align*}
C_1 + \epsilon B &\subset \{ x \mid a'x > b \}, \\
C_2 + \epsilon B &\subset \{ x \mid a'x < b \},
\end{align*}
\]

where \( B \) denotes the closed unit ball. Since \( a \neq 0 \), we also have

\[
\inf \{ a'y \mid y \in B \} < 0, \quad \sup \{ a'y \mid y \in B \} > 0.
\]

Therefore, it follows from the preceding relations that

\[
\begin{align*}
b &\leq \inf \{ a'x + \epsilon a'y \mid x \in C_1, y \in B \} < \inf \{ a'x \mid x \in C_1 \}, \\
b &\geq \sup \{ a'x + \epsilon a'y \mid x \in C_2, y \in B \} > \sup \{ a'x \mid x \in C_2 \}.
\end{align*}
\]

Thus, there exists a vector \( a \in \mathbb{R}^n \) such that

\[
\inf_{x \in C_1} a'x > \sup_{x \in C_2} a'x,
\]

proving (ii).

Next, we show that (ii) implies (iii). Suppose that (ii) holds, i.e., there exists some vector \( a \in \mathbb{R}^n \) such that

\[
\inf_{x \in C_1} a'x > \sup_{x \in C_2} a'x,
\]

Using the Schwartz inequality, we see that

\[
\begin{align*}
0 &< \inf_{x \in C_1} a'x - \sup_{x \in C_2} a'x \\
&= \inf_{x_1 \in C_1, x_2 \in C_2} a'(x_1 - x_2), \\
&\leq \inf_{x_1 \in C_1, x_2 \in C_2} \|a\| \|x_1 - x_2\|.
\end{align*}
\]

It follows that

\[
\inf_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\| > 0,
\]

thus proving (iii). Finally, we show that (iii) implies (i). If (iii) holds, we have for some \( \epsilon > 0 \),

\[
\inf_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\| > 2\epsilon > 0.
\]
From this we obtain for all $x_1 \in C_1$, all $x_2 \in C_2$, and for all $y_1$, $y_2$ with $\|y_1\| \leq \epsilon$, $\|y_2\| \leq \epsilon$,
\[
\|(x_1 + y_1) - (x_2 + y_2)\| \geq \|x_1 - x_2\| - \|y_1\| - \|y_2\| > 0,
\]
which implies that $0 \notin (C_1 + \epsilon B) - (C_2 + \epsilon B)$. Therefore, the convex sets $C_1 + \epsilon B$ and $C_2 + \epsilon B$ are disjoint. By the Separating Hyperplane Theorem, we see that $C_1 + \epsilon B$ and $C_2 + \epsilon B$ can be separated, i.e., $C_1 + \epsilon B$ and $C_2 + \epsilon B$ lie in opposite closed halfspaces associated with the hyperplane that separates them. Then, the sets $C_1 + (\epsilon/2)B$ and $C_2 + (\epsilon/2)B$ lie in opposite open halfspaces, which by definition implies that $C_1$ and $C_2$ can be separated strongly.

(b) Since $C_1$ and $C_2$ are disjoint, we have $0 \notin (C_1 - C_2)$. Any one of conditions (2)-(5) of Prop. 1.5.3 imply condition (1) of that proposition, which states that the set $C_1 - C_2$ is closed, i.e.,
\[
cl(C_1 - C_2) = C_1 - C_2.
\]
Hence, we have $0 \notin cl(C_1 - C_2)$, which implies that
\[
\inf_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\| > 0.
\]
From part (a), it follows that there exists a hyperplane separating $C_1$ and $C_2$ strongly.
Problem 4

We say that a function \( f : \mathbb{R}^n \to (-\infty, \infty] \) is \textit{quasiconvex} if all its level sets

\[
V_\gamma = \{ x \mid f(x) \leq \gamma \}
\]

are convex. Let \( X \) be a convex subset of \( \mathbb{R}^n \), let \( f \) be a quasiconvex function such that \( X \cap \text{dom}(f) \neq \emptyset \), and denote \( f^* = \inf_{x \in X} f(x) \).

(a) Assume that \( f \) is not constant on any line segment of \( X \), i.e., we do not have \( f(x) = c \) for some scalar \( c \) and all \( x \) in the line segment connecting any two distinct points of \( X \). Show that every local minimum of \( f \) over \( X \) is also global.

(b) Assume that \( X \) is closed, and \( f \) is closed and proper. Let \( \Gamma \) be the set of all \( \gamma > f^* \), and denote

\[
R_f = \bigcap_{\gamma \in \Gamma} R_\gamma, \quad L_f = \bigcap_{\gamma \in \Gamma} L_\gamma,
\]

where \( R_\gamma \) and \( L_\gamma \) are the recession cone and the lineality space of \( V_\gamma \), respectively. Use the line of proof of Prop. 3.2.4 to show that \( f \) attains a minimum over \( X \) if any one of the following conditions holds:

1. \( R_X \cap R_f = L_X \cap L_f \).
2. \( R_X \cap R_f \subset L_f \), and \( X \) is a polyhedral set.

Solution.

(a) Let \( x^* \) be a local minimum of \( f \) over \( X \) and assume, to arrive at a contradiction, that there exists a vector \( \bar{x} \in X \) such that \( f(\bar{x}) < f(x^*) \). Then, \( \bar{x} \) and \( x^* \) belong to the set \( X \cap V_{\gamma^*} \), where \( \gamma^* = f(x^*) \). Since this set is convex, the line segment connecting \( x^* \) and \( \bar{x} \) belongs to the set, implying that

\[
f(\alpha \bar{x} + (1 - \alpha)x^*) \leq \gamma^* = f(x^*), \quad \forall \alpha \in [0, 1].
\]

For each integer \( k \geq 1 \), there must exist an \( \alpha_k \in (0, 1/k] \) such that

\[
f(\alpha_k \bar{x} + (1 - \alpha_k)x^*) < f(x^*), \quad \text{for some } \alpha_k \in (0, 1/k]
\]

otherwise, we would have that \( f(x) \) is constant for \( x \) on the line segment connecting \( x^* \) and \( (1/k) \bar{x} + (1 - (1/k)) x^* \). This contradicts the local optimality of \( x^* \).

(b) We consider the level sets

\[
V_\gamma = \{ x \mid f(x) \leq \gamma \}
\]

for \( \gamma > f^* \). Let \( \{ \gamma_k \} \) be a scalar sequence such that \( \gamma_k \downarrow f^* \). Using the fact that for two nonempty closed convex sets \( C \) and \( D \) such that \( C \subset D \), we have \( R_C \subset R_D \), it can be seen that

\[
R_f = \bigcap_{\gamma \in \Gamma} R_\gamma = \bigcap_{k=1}^\infty R_{\gamma_k}.
\]

Similarly, \( L_f \) can be written as

\[
L_f = \bigcap_{\gamma \in \Gamma} L_\gamma = \bigcap_{k=1}^\infty L_{\gamma_k}.
\]

Under each of the conditions (1) and (2), we will show that the set of minima of \( f \) over \( X \), which is given by

\[
X^* = \bigcap_{k=1}^\infty (X \cap V_{\gamma_k})
\]

is nonempty.

Let condition (1) hold. The sets \( X \cap V_{\gamma_k} \) are nonempty, closed, convex, and nested. Furthermore, for each \( k \), their recession cone is given by \( R_X \cap R_{\gamma_k} \) and their lineality space is given by \( L_X \cap L_{\gamma_k} \). We have that

\[
\bigcap_{k=1}^\infty (R_X \cap R_{\gamma_k}) = R_X \cap R_f,
\]

and
and

$$\bigcap_{k=1}^{\infty} (L_X \cap L_{\gamma_k}) = L_X \cap L_f,$$

while by assumption $R_X \cap R_f = L_X \cap L_f$. Then it follows that $X^*$ is nonempty.

Let condition (2) hold. The sets $V_{\gamma_k}$ are nested and the intersection $X \cap V_{\gamma_k}$ is nonempty for all $k$. We also have by assumption that $R_X \cap R_f \in L_f$ and $X$ is a polyhedral set. It follows that $X^*$ is nonempty.
Problem 5

Let $F : \mathbb{R}^{n+m} \to (-\infty, \infty]$ be a closed proper convex function of two vectors $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, and let

$$X = \left\{ x \mid \inf_{z \in \mathbb{R}^m} F(x, z) < \infty \right\}.$$ 

Assume that the function $F(x, \cdot)$ is closed for each $x \in X$. Show that:

(a) If for some $\bar{x} \in X$, the minimum of $F(\bar{x}, \cdot)$ over $\mathbb{R}^m$ is attained at a nonempty and compact set, the same is true for all $x \in X$.

(b) If the functions $F(x, \cdot)$ are differentiable for all $x \in X$, they have the same asymptotic slopes along all directions, i.e., for each $d \in \mathbb{R}^m$, the value of $\lim_{\alpha \to \infty} \nabla_z F(x, z + \alpha d)'d$ is the same for all $x \in X$ and $z \in \mathbb{R}^m$.

Solution.

The recession cone of $F$ has the form

$$R_F = \{(d_x, d_z) \mid (d_x, d_z, 0) \in R_{epi}(F)\}.$$ 

The (common) recession cone of the nonempty level sets of $F(x, \cdot)$, $x \in X$, has the form

$$\{d_z \mid (0, d_z) \in R_F\},$$

for all $x \in X$, where $R_F$ is the recession cone of $F$. Furthermore, the recession function of $F(x, \cdot)$ is the same for all $x \in X$.

(a) By the compactness hypothesis, the recession cone of $F(\bar{x}, \cdot)$ consists of just the origin, so the same is true for the recession cones of all $F(x, \cdot)$, $x \in X$. Thus the nonempty level sets of $F(x, \cdot)$, $x \in X$, are all compact.

(b) This is a consequence of the fact that the recession function of $F(x, \cdot)$ is the same for all $x \in X$, and the comments following Prop. 1.4.5