Problem 1

Consider the convex programming problem

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in X, \quad g(x) \leq 0,
\end{align*}$$

of Section 5.3, and assume that the set $X$ is described by equality and inequality constraints as

$$X = \{ x \mid h_i(x) = 0, \ i = 1, \ldots, m, \ g_j(x) \leq 0, \ j = r + 1, \ldots, \bar{r} \}.$$  

Then the problem can alternatively be described without an abstract set constraint, in terms of all of the constraint functions

$$h_i(x) = 0, \ i = 1, \ldots, m, \quad g_j(x) \leq 0, \ j = 1, \ldots, \bar{r}.$$  

We call this the **extended representation** of (P). Show if there is no duality gap and there exists a dual optimal solution for the extended representation, the same is true for the original problem.

**Solution.**

Assume that there exists a dual optimal solution in the extended representation. Thus there exist nonnegative scalars $\lambda_1^*, \ldots, \lambda_m^*, \lambda_{m+1}^*, \ldots, \lambda_{\bar{m}}^*$ and $\mu_1^*, \ldots, \mu_r^*, \mu_{r+1}^*, \ldots, \mu_{\bar{r}}^*$ such that

$$f^* = \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i^* h_i(x) + \sum_{j=1}^{\bar{r}} \mu_j^* g_j(x) \right\},$$

from which we have

$$f^* \leq f(x) + \sum_{i=1}^{m} \lambda_i^* h_i(x) + \sum_{j=1}^{\bar{r}} \mu_j^* g_j(x), \quad \forall \ x \in \mathbb{R}^n.$$

For any $x \in X$, we have $h_i(x) = 0$ for all $i = 1, \ldots, m$, and $g_j(x) \leq 0$ for all $j = r + 1, \ldots, \bar{r}$, so that $\mu_j^* g_j(x) \leq 0$ for all $j = r + 1, \ldots, \bar{r}$. Therefore, it follows from the preceding relation that

$$f^* \leq f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \quad \forall \ x \in X.$$
Taking the infimum over all \( x \in X \), it follows that
\[
 f^* \leq \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \\
\leq \inf_{x \in X, \ g_j(x) \leq 0, \ j=1, \ldots, r} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \\
\leq \inf_{x \in X, \ h_i(x) = 0, \ \ i=1, \ldots, m} \ inf_{j=1, \ldots, r} \ f(x) \\
= f^*.
\]

Hence, equality holds throughout above, showing that the scalars \( \lambda_1^*, \ldots, \lambda_m^*, \mu_1^*, \ldots, \mu_r^* \) constitute a dual optimal solution for the original representation.

**Problem 2**

Consider the class of problems
\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in X, \quad g_j(x) \leq u_j, \quad j = 1, \ldots, r,
\end{align*}
\]
where \( u = (u_1, \ldots, u_r) \) is a vector parameterizing the right-hand side of the constraints. Given two distinct values \( \bar{u} \) and \( \tilde{u} \) of \( u \), let \( \bar{f} \) and \( \tilde{f} \) be the corresponding optimal values, and assume that \( \bar{f} \) and \( \tilde{f} \) are finite. Assume further that \( \bar{\mu} \) and \( \tilde{\mu} \) are corresponding dual optimal solutions and that there is no duality gap. Show that
\[
\bar{\mu}'(\bar{u} - \bar{u}) \leq \bar{f} - \tilde{f} \leq \tilde{\mu}'(\tilde{u} - \bar{u}).
\]

**Solution.**

We have
\[
\bar{f} = \inf_{x \in X} \{ f(x) + \bar{\mu}'(g(x) - \bar{u}) \},
\]
\[
f = \inf_{x \in X} \{ f(x) + \mu'(g(x) - u) \}.
\]

Let \( \bar{q}(\mu) \) denote the dual function of the problem corresponding to \( \bar{u} \):
\[
\bar{q}(\mu) = \inf_{x \in X} \{ f(x) + \mu'(g(x) - \bar{u}) \}.
\]

We have
\[
\bar{f} - f = \inf_{x \in X} \{ f(x) + \bar{\mu}'(g(x) - \bar{u}) \} - \inf_{x \in X} \{ f(x) + \mu'(g(x) - u) \}
\]
\[
= \inf_{x \in X} \{ f(x) + \bar{\mu}'(g(x) - \bar{u}) \} - \inf_{x \in X} \{ f(x) + \mu'(g(x) - \bar{u}) \} + \mu'(u - \bar{u})
\]
\[
= \bar{q}(\bar{\mu}) - \bar{q}(\mu) + \mu'(u - \bar{u})
\]
\[
\geq \mu'(u - \bar{u}),
\]
where the last inequality holds because \( \bar{\mu} \) maximizes \( \bar{q} \).

This proves the left-hand side of the desired inequality. Interchanging the roles of \( \bar{f}, \bar{u}, \bar{\mu}, \) and \( f, u, \mu, \) shows the desired right-hand side.
Problem 3

Let $g_j : \mathbb{R}^n \mapsto \mathbb{R}$, $j = 1, \ldots, r$, be convex functions over the nonempty convex subset of $\mathbb{R}^n$. Show that the system

$$g_j(x) < 0, \quad j = 1, \ldots, r,$$

has no solution within $X$ if and only if there exists a vector $\mu \in \mathbb{R}^r$ such that

$$\sum_{j=1}^r \mu_j = 1, \quad \mu \geq 0,$$

$$\mu' g(x) \geq 0, \quad \forall \ x \in X.$$

*Hint:* Consider the convex program

$$\begin{align*}
\text{minimize} & \quad y \\
\text{subject to} & \quad x \in X, \quad y \in \mathbb{R}, \quad g_j(x) \leq y, \quad j = 1, \ldots, r.
\end{align*}$$

**Solution.**

The dual function for the problem in the hint is

$$q(\mu) = \inf_{y \in \mathbb{R}, x \in X} \left\{ y + \sum_{j=1}^r \mu_j (g_j(x) - y) \right\}$$

$$= \begin{cases} 
\inf_{x \in X} \sum_{j=1}^r \mu_j g_j(x) & \text{if } \sum_{j=1}^r \mu_j = 1, \\
-\infty & \text{if } \sum_{j=1}^r \mu_j \neq 1.
\end{cases}$$

The problem in the hint satisfies the Slater condition, so the dual problem has an optimal solution $\mu^*$ and there is no duality gap.

Clearly the problem in the hint has an optimal value that is greater or equal to 0 if and only if the system of inequalities

$$g_j(x) < 0, \quad j = 1, \ldots, r,$$

has no solution within $X$. Since there is no duality gap, we have

$$\max_{\mu \geq 0, \sum_{j=1}^r \mu_j = 1} q(\mu) \geq 0$$

if and only if the system of inequalities $g_j(x) < 0, \ j = 1, \ldots, r$, has no solution within $X$. This is equivalent to the statement we want to prove.

Problem 4

Consider the problem

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in X, \quad g(x) \leq 0,
\end{align*}$$

where $X$ is a convex set, and $f$ and $g_j$ are convex over $X$. Assume that the problem has at least one feasible solution. Show that the following are equivalent.

(i) The dual optimal value $q^* = \sup_{\mu \in \mathbb{R}^r} q(\mu)$ is finite.

(ii) The primal function $p$ is proper.
(iii) The set
\[ M = \{(u, w) \in \mathbb{R}^{r+1} \mid \text{there is an } x \in X \text{ such that } g(x) \leq u, f(x) \leq w\} \]
does not contain a vertical line.

Solution.
We note that \(-q\) is closed and convex, and that
\[ q(\mu) = \inf_{u \in \mathbb{R}^r} \{p(u) + \mu'u\}, \quad \forall \mu \in \mathbb{R}^r. \]
Since \(q(\mu) \leq p(0)\) for all \(\mu \in \mathbb{R}^r\), given the feasibility of the problem [i.e., \(p(0) < \infty\)], we see that \(q^*\) is finite if and only if \(q\) is proper. Since \(q\) is the conjugate of \(p(-u)\) and \(p\) is convex, by the Conjugacy Theorem, \(q\) is proper if and only if \(p\) is proper. Hence (i) is equivalent to (ii).

We note that the epigraph of \(p\) is the closure of \(M\). Hence, given the feasibility of the problem, (ii) is equivalent to the closure of \(M\) not containing a vertical line. Since \(M\) is convex, its closure does not contain a line if and only if \(M\) does not contain a line (since the closure and the relative interior of \(M\) have the same recession cone). Hence (ii) is equivalent to (iii).

Problem 5
Consider a proper convex function \(F\) of two vectors \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}^m\). For a fixed \((\bar{x}, \bar{y}) \in \text{dom}(F)\), let \(\partial_x F(\bar{x}, \bar{y})\) and \(\partial_y F(\bar{x}, \bar{y})\) be the subdifferentials of the functions \(F(\cdot, \bar{y})\) and \(F(\bar{x}, \cdot)\) at \(\bar{x}\) and \(\bar{y}\), respectively. (a) Show that
\[ \partial F(\bar{x}, \bar{y}) \subset \partial_x F(\bar{x}, \bar{y}) \times \partial_y F(\bar{x}, \bar{y}), \]
and give an example showing that the inclusion may be strict in general. (b) Assume that \(F\) has the form
\[ F(x, y) = h_1(x) + h_2(y) + h(x, y), \]
where \(h_1\) and \(h_2\) are proper convex functions, and \(h\) is convex, real-valued, and differentiable. Show that the formula of part (a) holds with equality.

Solution.
(a) We have \((g_x, g_y) \in \partial F(\bar{x}, \bar{y})\) if and only if
\[ F(x, y) \geq F(\bar{x}, \bar{y}) + g_x(x - \bar{x}) + g_y(y - \bar{y}), \quad \forall x \in \mathbb{R}^n, \ y \in \mathbb{R}^m. \]
By setting \(y = \bar{y}\), we obtain that \(g_x \in \partial_x F(\bar{x}, \bar{y})\), and by setting \(x = \bar{x}\), we obtain that \(g_y \in \partial_y F(\bar{x}, \bar{y})\), so that \((g_x, g_y) \in \partial_x F(\bar{x}, \bar{y}) \times \partial_y F(\bar{x}, \bar{y})\).

For an example where the inclusion is strict, consider any function whose subdifferential is not a Cartesian product at some point, such as \(F(x, y) = |x + y|\) at points \((\bar{x}, \bar{y})\) with \(\bar{x} + \bar{y} = 0\).
(b) Since \(F\) is the sum of functions of the given form, we have
\[ \partial F(\bar{x}, \bar{y}) = \{(g_x, 0) \mid g_x \in \partial h_1(\bar{x})\} + \{(0, g_y) \mid g_y \in \partial h_2(\bar{y})\} + \{\nabla h(\bar{x}, \bar{y})\} \]
[the relative interior condition of the proposition is clearly satisfied]. Since
\[ \nabla h(\bar{x}, \bar{y}) = (\nabla_x h(\bar{x}, \bar{y}), \nabla_y h(\bar{x}, \bar{y})), \]
\[ \partial_x F(\bar{x}, \bar{y}) = \partial h_1(\bar{x}) + \nabla_x h(\bar{x}, \bar{y}), \]
\[ \partial_y F(\bar{x}, \bar{y}) = \partial h_2(\bar{y}) + \nabla_y h(\bar{x}, \bar{y}), \]
the result follows.
Problem 6

This exercise shows how a duality gap results in nondifferentiability of the dual function. Consider the problem

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in X, \quad g(x) \leq 0,
\end{align*}$$

and assume that for all $\mu \geq 0$, the infimum of the Lagrangian $L(x, \mu)$ over $X$ is attained by at least one $x_\mu \in X$. Show that if there is a duality gap, then the dual function $q(\mu) = \inf_{x \in X} L(x, \mu)$ is nondifferentiable at every dual optimal solution. Hint: If $q$ is differentiable at a dual optimal solution $\mu^*$, by the theory of Section 5.3, we must have $\partial q(\mu^*)/\partial \mu_j \leq 0$ and $\mu_j^* \partial q(\mu^*)/\partial \mu_j = 0$ for all $j$. Use optimality conditions for $\mu^*$, together with any vector $x_{\mu^*}$ that minimizes $L(x, \mu^*)$ over $X$, to show that there is no duality gap.

Solution.

To obtain a contradiction, assume that $q$ is differentiable at some dual optimal solution $\mu^* \in M$, where $M = \{ \mu \in R^r \mid \mu \geq 0 \}$. Then

$$\nabla q(\mu^*)(\mu^* - \mu) \geq 0, \quad \forall \mu \geq 0.$$ 

If $\mu_j^* = 0$, then by letting $\mu = \mu^* + \gamma e_j$ for a scalar $\gamma \geq 0$, and the vector $e_j$ whose $j$th component is 1 and the other components are 0, from the preceding relation we obtain $\partial q(\mu^*)/\partial \mu_j \leq 0$. Similarly, if $\mu_j^* > 0$, then by letting $\mu = \mu^* + \gamma e_j$ for a sufficiently small scalar $\gamma$ (small enough so that $\mu^* + \gamma e_j \in M$), from the preceding relation we obtain $\partial q(\mu^*)/\partial \mu_j = 0$. Hence

$$\partial q(\mu^*)/\partial \mu_j \leq 0, \quad \forall j = 1, \ldots, r,$$

$$\mu_j^* \partial q(\mu^*)/\partial \mu_j = 0, \quad \forall j = 1, \ldots, r.$$

Since $q$ is differentiable at $\mu^*$, we have that

$$\nabla q(\mu^*) = g(x^*),$$

for some vector $x^* \in X$ such that $q(\mu^*) = L(x^*, \mu^*)$. This and the preceding two relations imply that $x^*$ and $\mu^*$ satisfy the necessary and sufficient optimality conditions for an optimal primal and dual optimal solution pair. It follows that there is no duality gap, a contradiction.

Problem 7

Consider the problem

$$\begin{align*}
\text{minimize} & \quad f(x) = 10x_1 + 3x_2 \\
\text{subject to} & \quad 5x_1 + x_2 \geq 4, x_1, x_2 = 0 \text{ or } 1,
\end{align*}$$

(a) Sketch the set of constraint-cost pairs $\{(4 - 5x_1 - x_2, 10x_1 + 3x_2)|x_1, x_2 = 0 \text{ or } 1\}$.
(b) Describe the corresponding MC/MC framework as per Section 4.2.3.
(c) Solve the problem and its dual, and relate the solutions to your sketch in part (a).

Solution.

(a) The set of constraint-cost pairs contains 4 points: (-2,13), (-1,10), (3,3), (4,0).
(b) To each of these 4 points we add the first orphan and we get the $M$ set.
(c) The primal optimal solution is $x^* = (1, 0)$ and the primal optimal cost is $p^* = 10$. The dual function is easily found to be:

$$q(\mu) = \begin{cases} 
4\mu & \text{if } \mu \leq 2, \\
10 - \mu & \text{if } 2 \leq \mu \leq 3, \\
13 - 2\mu & \text{if } 3 \leq \mu.
\end{cases}$$

Therefore $q^* = 8$. This is the intersection of the line segment connecting the points $(4, 0), (-1, 10)$ with the y-axis.
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