LECTURE 7

LECTURE OUTLINE

• Review of hyperplane separation
• Nonvertical hyperplanes
• Convex conjugate functions
• Conjugacy theorem
• Examples

Reading: Section 1.5, 1.6
**ADDITIONAL THEOREMS**

- **Fundamental Characterization:** The closure of the convex hull of a set \( C \subset \mathbb{R}^n \) is the intersection of the closed halfspaces that contain \( C \). (Proof uses the strict separation theorem.)

- We say that a hyperplane *properly separates* \( C_1 \) and \( C_2 \) if it separates \( C_1 \) and \( C_2 \) and does not fully contain both \( C_1 \) and \( C_2 \).

- **Proper Separation Theorem:** Let \( C_1 \) and \( C_2 \) be two nonempty convex subsets of \( \mathbb{R}^n \). There exists a hyperplane that properly separates \( C_1 \) and \( C_2 \) if and only if

\[
\text{ri}(C_1) \cap \text{ri}(C_2) = \emptyset
\]
PROPER POLYHEDRAL SEPARATION

• Recall that two convex sets $C$ and $P$ such that

$$\text{ri}(C) \cap \text{ri}(P) = \emptyset$$

can be properly separated, i.e., by a hyperplane that does not contain both $C$ and $P$.

• If $P$ is polyhedral and the slightly stronger condition

$$\text{ri}(C) \cap P = \emptyset$$

holds, then the properly separating hyperplane can be chosen so that it does not contain the non-polyhedral set $C$ while it may contain $P$.

On the left, the separating hyperplane can be chosen so that it does not contain $C$. On the right where $P$ is not polyhedral, this is not possible.
NONVERTICAL HYPERPLANES

• A hyperplane in $\mathbb{R}^{n+1}$ with normal $(\mu, \beta)$ is nonvertical if $\beta \neq 0$.

• It intersects the $(n+1)$st axis at $\xi = (\mu/\beta)\overline{u} + \overline{w}$, where $(\overline{u}, \overline{w})$ is any vector on the hyperplane.

A nonvertical hyperplane that contains the epigraph of a function in its “upper” halfspace, provides lower bounds to the function values.

• The epigraph of a proper convex function does not contain a vertical line, so it appears plausible that it is contained in the “upper” halfspace of some nonvertical hyperplane.
NONVERTICAL HYPERPLANE THEOREM

- Let $C$ be a nonempty convex subset of $\mathbb{R}^{n+1}$ that contains no vertical lines. Then:

  (a) $C$ is contained in a closed halfspace of a nonvertical hyperplane, i.e., there exist $\mu \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ with $\beta \neq 0$, and $\gamma \in \mathbb{R}$ such that $\mu'u + \beta w \geq \gamma$ for all $(u, w) \in C$.

  (b) If $(\bar{u}, \bar{w}) \notin \text{cl}(C)$, there exists a nonvertical hyperplane strictly separating $(\bar{u}, \bar{w})$ and $C$.

**Proof:** Note that $\text{cl}(C)$ contains no vert. line [since $C$ contains no vert. line, $\text{ri}(C)$ contains no vert. line, and $\text{ri}(C)$ and $\text{cl}(C)$ have the same recession cone]. So we just consider the case: $C$ closed.

(a) $C$ is the intersection of the closed halfspaces containing $C$. If all these corresponded to vertical hyperplanes, $C$ would contain a vertical line.

(b) There is a hyperplane strictly separating $(\bar{u}, \bar{w})$ and $C$. If it is nonvertical, we are done, so assume it is vertical. “Add” to this vertical hyperplane a small $\epsilon$-multiple of a nonvertical hyperplane containing $C$ in one of its halfspaces as per (a).
CONJUGATE CONVEX FUNCTIONS

• Consider a function $f$ and its epigraph

Nonvertical hyperplanes supporting $\text{epi}(f)$

$\iff$ Crossing points of vertical axis

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{x'y - f(x)\}, \quad y \in \mathbb{R}^n.$$  

• For any $f : \mathbb{R}^n \mapsto [-\infty, \infty]$, its conjugate convex function is defined by

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{x'y - f(x)\}, \quad y \in \mathbb{R}^n$$
EXAMPLES

\[ f^*(y) = \sup_{x \in \mathbb{R}^n} \{ x' y - f(x) \}, \quad y \in \mathbb{R}^n \]
CONJUGATE OF CONJUGATE

- From the definition
  \[ f^*(y) = \sup_{x \in \mathbb{R}^n} \left\{ x'y - f(x) \right\}, \quad y \in \mathbb{R}^n, \]

  note that \( f^* \) is convex and closed.

- Reason: \( \text{epi}(f^*) \) is the intersection of the epigraphs of the linear functions of \( y \)
  \[ x'y - f(x) \]
  as \( x \) ranges over \( \mathbb{R}^n \).

- Consider the conjugate of the conjugate:
  \[ f^{**}(x) = \sup_{y \in \mathbb{R}^n} \left\{ y'x - f^*(y) \right\}, \quad x \in \mathbb{R}^n. \]

  \( f^{**} \) is convex and closed.

- Important fact/Conjugacy theorem: If \( f \) is closed proper convex, then \( f^{**} = f \).
CONJUGACY THEOREM - VISUALIZATION

\[ f^*(y) = \sup_{x \in \mathbb{R}^n} \{ x' y - f(x) \}, \quad y \in \mathbb{R}^n \]

\[ f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{ y' x - f^*(y) \}, \quad x \in \mathbb{R}^n \]

- If \( f \) is closed convex proper, then \( f^{**} = f \).
CONJUGACY THEOREM

• Let \( f : \mathbb{R}^n \mapsto (-\infty, \infty] \) be a function, let \( \text{cl} \ f \) be its convex closure, let \( f^* \) be its convex conjugate, and consider the conjugate of \( f^* \),

\[
f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{ y'x - f^*(y) \}, \quad x \in \mathbb{R}^n
\]

(a) We have

\[
f(x) \geq f^{**}(x), \quad \forall \ x \in \mathbb{R}^n
\]

(b) If \( f \) is convex, then properness of any one of \( f, f^*, \) and \( f^{**} \) implies properness of the other two.

(c) If \( f \) is closed proper and convex, then

\[
f(x) = f^{**}(x), \quad \forall \ x \in \mathbb{R}^n
\]

(d) If \( \text{cl} \ f(x) > -\infty \) for all \( x \in \mathbb{R}^n \), then

\[
\text{cl} \ f(x) = f^{**}(x), \quad \forall \ x \in \mathbb{R}^n
\]
PROOF OF CONJUGACY THEOREM (A), (C)

• (a) For all \( x, y \), we have \( f^*(y) \geq y'x - f(x) \), implying that \( f(x) \geq \sup_y \{y'x - f^*(y)\} = f^{**}(x) \).

• (c) By contradiction. Assume there is \( (x, \gamma) \in \text{epi}(f^{**}) \) with \( (x, \gamma) \notin \text{epi}(f) \). There exists a non-vertical hyperplane with normal \( (y, -1) \) that strictly separates \( (x, \gamma) \) and \( \text{epi}(f) \). (The vertical component of the normal vector is normalized to -1.)

• Consider two parallel hyperplanes, translated to pass through \( (x, f(x)) \) and \( (x, f^{**}(x)) \). Their vertical crossing points are \( x'y - f(x) \) and \( x'y - f^{**}(x) \), and lie strictly above and below the crossing point of the strictly sep. hyperplane. Hence

\[
x'y - f(x) > x'y - f^{**}(x)
\]

the fact \( f \geq f^{**} \). Q.E.D.
A COUNTEREXAMPLE

• A counterexample (with closed convex but improper $f$) showing the need to assume properness in order for $f = f^{**}$:

$$f(x) = \begin{cases} 
\infty & \text{if } x > 0, \\
-\infty & \text{if } x \leq 0.
\end{cases}$$

We have

$$f^*(y) = \infty, \quad \forall \ y \in \mathbb{R}^n,$$

$$f^{**}(x) = -\infty, \quad \forall \ x \in \mathbb{R}^n.$$

But

$$\text{cl} f = f,$$

so $\text{cl} f \neq f^{**}$. 