6.254 : Game Theory with Engineering Applications
Lecture 5: Existence of a Nash Equilibrium

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Outline

- Pricing-Congestion Game Example
- Existence of a Mixed Strategy Nash Equilibrium in Finite Games
- Existence in Games with Infinite Strategy Spaces

Reading:
- Fudenberg and Tirole, Chapter 1.
Introduction

- In this lecture, we study the question of existence of a Nash equilibrium in both games with finite and infinite pure strategy spaces.
- We start with an example, pricing-congestion game, where players have infinitely many pure strategies.
- We consider two instances of this game, one of which has a unique pure Nash equilibrium, and the other does not have any pure Nash equilibria.
Pricing-Congestion Game

Consider a price competition model studied in [Acemoglu and Ozdaglar 07].

Consider a parallel link network with $I$ links. Assume that $d$ units of flow is to be routed through this network. We assume that this flow is the aggregate flow of many infinitesimal users.

Let $l_i(x_i)$ denote the latency function of link $i$, which represents the delay or congestion costs as a function of the total flow $x_i$ on link $i$.

Assume that the links are owned by independent providers. Provider $i$ sets a price $p_i$ per unit of flow on link $i$.

The effective cost of using link $i$ is $p_i + l_i(x_i)$.

Users have a reservation utility equal to $R$, i.e., if $p_i + l_i(x_i) > R$, then no traffic will be routed on link $i$. 
Example 1

- We consider an example with two links and latency functions $l_1(x_1) = 0$ and $l_2(x_2) = \frac{3x_2}{2}$. For simplicity, we assume that $R = 1$ and $d = 1$.
- Given the prices $(p_1, p_2)$, we assume that the flow is allocated according to Wardrop equilibrium, i.e., the flows are routed along minimum effective cost paths and the effective cost cannot exceed the reservation utility.

Definition

A flow vector $x = [x_i]_{i=1,...,l}$ is a Wardrop equilibrium if $\sum_{i=1}^l x_i \leq d$ and

$$p_i + l_i(x_i) = \min_j \{p_j + l_j(x_j)\}, \quad \text{for all } i \text{ with } x_i > 0,$$

$$p_i + l_i(x_i) \leq R, \quad \text{for all } i \text{ with } x_i > 0,$$

with $\sum_{i=1}^l x_i = d$ if $\min_j \{p_j + l_j(x_j)\} < R$. 
Example 1 (Continued)

- We use the preceding characterization to determine the flow allocation on each link given prices $0 \leq p_1, p_2 \leq 1$:

  $$x_2(p_1, p_2) = \begin{cases} \frac{2}{3}(p_1 - p_2), & p_1 \geq p_2, \\ 0, & \text{otherwise}, \end{cases}$$

  and $x_1(p_1, p_2) = 1 - x_2(p_1, p_2)$.

- The payoffs for the providers are given by:

  $$u_1(p_1, p_2) = p_1 \times x_1(p_1, p_2)$$
  $$u_2(p_1, p_2) = p_2 \times x_2(p_1, p_2)$$

- We find the pure strategy Nash equilibria of this game by characterizing the best response correspondences, $B_i(p_{-i})$ for each player.

  - The following analysis assumes that at the Nash equilibria $(p_1, p_2)$ of the game, the corresponding Wardrop equilibria $x$ satisfies $x_1 > 0$, $x_2 > 0$, and $x_1 + x_2 = 1$. For the proofs of these statements, see [Acemoglu and Ozdaglar 07].
Example 1 (Continued)

- In particular, for a given \( p_2 \), \( B_1(p_2) \) is the optimal solution set of the following optimization problem

\[
\text{maximize } \quad 0 \leq p_1 \leq 1, \quad 0 \leq x_1 \leq 1 \quad \quad p_1x_1
\]

\[
\text{subject to } \quad p_1 = p_2 + \frac{3}{2}(1 - x_1)
\]

- Solving the preceding optimization problem, we find that

\[
B_1(p_2) = \min \left\{ 1, \frac{3}{4} + \frac{p_2}{2} \right\}.
\]

Similarly, \( B_2(p_1) = \frac{p_1}{2} \).
Example 1 (Continued)

- The figure illustrates the best response correspondences as a function of $p_1$ and $p_2$. The correspondences intersect at the unique point $(p_1, p_2) = (1, \frac{1}{2})$, which is the unique pure strategy equilibrium.
Example 2

- We next consider a similar example with latency functions given by

\[ l_1(x) = 0, \quad l_2(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq 1/2 \\
\frac{x-1/2}{\varepsilon} & x \geq 1/2,
\end{cases} \]

for some sufficiently small \( \varepsilon > 0 \).

- The following list considers all candidate Nash equilibria \((p_1, p_2)\) and profitable unilateral deviations for \( \varepsilon \) sufficiently small, thus establishing the nonexistence of a pure strategy Nash equilibrium:

  - \( p_1 = p_2 = 0 \): A small increase in the price of provider 1 will generate positive profits, thus provider 1 has an incentive to deviate.
  - \( p_1 = p_2 > 0 \): Let \( x \) be the corresponding flow allocation. If \( x_1 = 1 \), then provider 2 has an incentive to decrease its price. If \( x_1 < 1 \), then provider 1 has an incentive to decrease its price.
  - \( 0 \leq p_1 < p_2 \): Player 1 has an incentive to increase its price since its flow allocation remains the same.
  - \( 0 \leq p_2 < p_1 \): For \( \varepsilon \) sufficiently small, the profit function of player 2, given \( p_1 \), is strictly increasing as a function of \( p_2 \), showing that provider 2 has an incentive to increase its price.
Existence Results

- We start by analyzing existence of a Nash equilibrium in finite (strategic form) games, i.e., games with finite strategy sets.

Theorem

(Nash) *Every finite game has a mixed strategy Nash equilibrium.*

- Implication: matching pennies game necessarily has a mixed strategy equilibrium.
- Why is this important?
  - Without knowing the existence of an equilibrium, it is difficult (perhaps meaningless) to try to understand its properties.
  - Armed with this theorem, we also know that every finite game has an equilibrium, and thus we can simply try to locate the equilibria.
Approach

- Recall that a mixed strategy profile $\sigma^*$ is a NE if
  \[ u_i(\sigma_i^*, \sigma^*_{-i}) \geq u_i(\sigma_i, \sigma^*_{-i}), \quad \text{for all } \sigma_i \in \Sigma_i. \]

- In other words, $\sigma^*$ is a NE if and only if $\sigma_i^* \in B^*_{-i}(\sigma^*_{-i})$ for all $i$, where $B^*_{-i}(\sigma^*_{-i})$ is the best response of player $i$, given that the other players’ strategies are $\sigma^*_{-i}$.

- We define the correspondence $B : \Sigma \rightrightarrows \Sigma$ such that for all $\sigma \in \Sigma$, we have
  \[ B(\sigma) = [B_i(\sigma_{-i})]_{i \in I} \quad (1) \]

- The existence of a Nash equilibrium is then equivalent to the existence of a mixed strategy $\sigma$ such that $\sigma \in B(\sigma)$: i.e., existence of a fixed point of the mapping $B$.

- We will establish existence of a Nash equilibrium in finite games using a fixed point theorem.
Definitions

- A set in a Euclidean space is compact if and only if it is bounded and closed.
- A set $S$ is **convex** if for any $x, y \in S$ and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in S$.

![Convex set](image1)

![Not a convex set](image2)

convex set

not a convex set
Weierstrass’s Theorem

Theorem

(Weierstrass) Let $A$ be a nonempty compact subset of a finite dimensional Euclidean space and let $f : A \to \mathbb{R}$ be a continuous function. Then there exists an optimal solution to the optimization problem

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in A.
\end{align*}$$

There exists no optimal $x$ that attains it.
Kakutani’s Fixed Point Theorem

Theorem

(Kakutani) Let $A$ be a non-empty subset of a finite dimensional Euclidean space. Let $f : A \rightrightarrows A$ be a correspondence, with $x \in A \mapsto f(x) \subseteq A$, satisfying the following conditions:

- $A$ is a compact and convex set.
- $f(x)$ is non-empty for all $x \in A$.
- $f(x)$ is a convex-valued correspondence: for all $x \in A$, $f(x)$ is a convex set.
- $f(x)$ has a closed graph: that is, if $\{x^n, y^n\} \rightarrow \{x, y\}$ with $y^n \in f(x^n)$, then $y \in f(x)$.

Then, $f$ has a fixed point, that is, there exists some $x \in A$, such that $x \in f(x)$. 

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Kakutani’s Fixed Point Theorem—Graphical Illustration

$f(x) = x$

$f(x)$ is not convex-valued

$f(x)$ does not have a closed graph
Proof of Nash’s Theorem

- The idea is to apply Kakutani’s theorem to the best response correspondence $B : \Sigma \rightrightarrows \Sigma$. We show that $B(\sigma)$ satisfies the conditions of Kakutani’s theorem.

- $\Sigma$ is compact, convex, and non-empty.
  - By definition
    \[ \Sigma = \prod_{i \in \mathcal{I}} \Sigma_i \]
    where each $\Sigma_i = \{x \mid \sum_j x_j = 1\}$ is a simplex of dimension $|S_i| - 1$, thus each $\Sigma_i$ is closed and bounded, and thus compact. Their product set is also compact.

- $B(\sigma)$ is non-empty.
  - By definition,
    \[ B_i(\sigma_{-i}) = \arg \max_{x \in \Sigma_i} u_i(x, \sigma_{-i}) \]
    where $\Sigma_i$ is non-empty and compact, and $u_i$ is linear in $x$. Hence, $u_i$ is continuous, and by Weistrass’s theorem $B(\sigma)$ is non-empty.
Proof (continued)

3. $B(\sigma)$ is a convex-valued correspondence.
   
   Equivalently, $B(\sigma) \subseteq \Sigma$ is convex if and only if $B_i(\sigma_{-i})$ is convex for all $i$. Let $\sigma'_i, \sigma''_i \in B_i(\sigma_{-i})$.
   
   Then, for all $\lambda \in [0, 1] \in B_i(\sigma_{-i})$, we have
   
   $$u_i(\sigma'_i, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \quad \text{for all } \tau_i \in \Sigma_i,$$
   $$u_i(\sigma''_i, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \quad \text{for all } \tau_i \in \Sigma_i.$$
   
   The preceding relations imply that for all $\lambda \in [0, 1]$, we have
   
   $$\lambda u_i(\sigma'_i, \sigma_{-i}) + (1 - \lambda) u_i(\sigma''_i, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \quad \text{for all } \tau_i \in \Sigma_i.$$

   By the linearity of $u_i$,
   
   $$u_i(\lambda \sigma'_i + (1 - \lambda) \sigma''_i, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \quad \text{for all } \tau_i \in \Sigma_i.$$

   Therefore, $\lambda \sigma'_i + (1 - \lambda) \sigma''_i \in B_i(\sigma_{-i})$, showing that $B(\sigma)$ is convex-valued.
Proof (continued)

4. \( B(\sigma) \) has a closed graph.
   
   - Suppose to obtain a contradiction, that \( B(\sigma) \) does not have a closed graph.
   - Then, there exists a sequence \((\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})\) with \( \hat{\sigma}^n \in B(\sigma^n) \), but \( \hat{\sigma} \notin B(\sigma) \), i.e., there exists some \( i \) such that \( \hat{\sigma}_i \notin B_i(\sigma_{-i}) \).
   - This implies that there exists some \( \sigma'_i \in \Sigma_i \) and some \( \epsilon > 0 \) such that
     \[
     u_i(\sigma'_i, \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon.
     \]
   
   - By the continuity of \( u_i \) and the fact that \( \sigma_{-i}^n \to \sigma_{-i} \), we have for sufficiently large \( n \),
     \[
     u_i(\sigma'_i, \sigma_{-i}^n) \geq u_i(\sigma'_i, \sigma_{-i}) - \epsilon.
     \]
Proof (continued)

[step 4 continued] Combining the preceding two relations, we obtain

\[ u_i(\sigma'_i, \sigma^n_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon \geq u_i(\hat{\sigma}^n_i, \sigma^n_{-i}) + \epsilon, \]

where the second relation follows from the continuity of \( u_i \). This contradicts the assumption that \( \hat{\sigma}^n_i \in B_i(\sigma^n_{-i}) \), and completes the proof.

- The existence of the fixed point then follows from Kakutani’s theorem.
- If \( \sigma^* \in B(\sigma^*) \), then by definition \( \sigma^* \) is a mixed strategy equilibrium.
Existence of Equilibria for Infinite Games

- A similar theorem to Nash’s existence theorem applies for pure strategy existence in infinite games.

**Theorem**

*(Debreu, Glicksberg, Fan)* Consider a strategic form game \( \langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle \) such that for each \( i \in \mathcal{I} \)

- \( S_i \) is compact and convex;
- \( u_i (s_i, s_{-i}) \) is continuous in \( s_{-i} \);
- \( u_i (s_i, s_{-i}) \) is continuous and concave in \( s_i \) [in fact quasi-concavity suffices].

Then a pure strategy Nash equilibrium exists.
Suppose $S$ is a convex set. Then a function $f : S \rightarrow \mathbb{R}$ is concave if for any $x, y \in S$ and any $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y).$$
Proof

• Now define the best response correspondence for player \( i \),
  \( B_i : S_{-i} \rightarrow S_i, \)

  \[ B_i(s_{-i}) = \{ s'_i \in S_i \mid u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_i \in S_i \} \ . \]

  Thus restriction to pure strategies.

• Define the set of best response correspondences as

  \[ B(s) = [B_i(s_{-i})]_{i \in \mathcal{I}} . \]

  and

  \[ B : S \rightarrow S. \]
Proof (continued)

- We will again apply Kakutani’s theorem to the best response correspondence $B : S \rightrightarrows S$ by showing that $B(s)$ satisfies the conditions of Kakutani’s theorem.

- $S$ is compact, convex, and non-empty.
  - By definition
    \[
    S = \prod_{i \in I} S_i
    \]
    since each $S_i$ is compact [convex, nonempty] and finite product of compact [convex, nonempty] sets is compact [convex, nonempty].

- $B(s)$ is non-empty.
  - By definition,
    \[
    B_i(s_{-i}) = \arg \max_{s \in S_i} u_i(s, s_{-i})
    \]
    where $S_i$ is non-empty and compact, and $u_i$ is continuous in $s$ by assumption. Then by Weistrass’s theorem $B(s)$ is non-empty.
**Proof (continued)**

3. \( B(s) \) is a convex-valued correspondence.

- This follows from the fact that \( u_i(s_i, s_{-i}) \) is concave [or quasi-concave] in \( s_i \). Suppose not, then there exists some \( i \) and some \( s_{-i} \in S_{-i} \) such that \( B_i(s_{-i}) \in \arg \max_{s \in S_i} u_i(s, s_{-i}) \) is not convex.
- This implies that there exists \( s_i', s_i'' \in S_i \) such that \( s_i', s_i'' \in B_i(s_{-i}) \) and \( \lambda s_i' + (1 - \lambda) s_i'' \not\in B_i(s_{-i}) \). In other words,

\[
\lambda u_i(s_i', s_{-i}) + (1 - \lambda) u_i(s_i'', s_{-i}) > u_i(\lambda s_i' + (1 - \lambda) s_i'', s_{-i})
\]

But this violates the concavity of \( u_i(s_i, s_{-i}) \) in \( s_i \) [recall that for a concave function \( f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y) \)].

- Therefore \( B(s) \) is convex-valued.

4. The proof that \( B(s) \) has a closed graph is identical to the previous proof.
Remarks

- Nash’s theorem is a special case of this theorem: Strategy spaces are simplices and utilities are linear in (mixed) strategies, hence they are concave functions of (mixed) strategies.

Continuity properties of the “Nash equilibrium set”:
- Consider strategic form games with finite pure strategy sets $S_i$ and utilities $u_i(s, \lambda)$, where $u_i$ is a continuous function of $\lambda$.
- Let $G(\lambda) = \langle I, (S_i), (u_i(s, \lambda)) \rangle$ and let $E(\lambda)$ denote the Nash correspondence that associates with each $\lambda$, the set of (mixed) Nash equilibria of $G(\lambda)$.

Proposition

Assume that $\lambda \in \Lambda$, where $\Lambda$ is a compact set. Then $E(\lambda)$ has a closed graph.

- Proof similar to the proof of closedness of $B(\sigma)$ in Nash’s theorem.
- This does not imply continuity of the Nash equilibrium set $E(\lambda)$!!
Existence of Nash Equilibria

- Can we relax (quasi)concavity?
- **Example:** Consider the game where two players pick a location $s_1, s_2 \in \mathbb{R}^2$ on the circle. The payoffs are

$$u_1(s_1, s_2) = -u_2(s_1, s_2) = d(s_1, s_2),$$

where $d(s_1, s_2)$ denotes the Euclidean distance between $s_1, s_2 \in \mathbb{R}^2$.

- No pure Nash equilibrium.
- However, it can be shown that the strategy profile where both mix uniformly on the circle is a mixed Nash equilibrium.
A More Powerful Theorem

Theorem

**Glicksberg** Consider a strategic form game \( (\mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}}) \) such that for each \( i \in \mathcal{I} \)

- \( S_i \) is a nonempty and compact metric space;
- \( u_i (s_i, s_{-i}) \) is continuous in \( s \).

Then a mixed strategy Nash equilibrium exists.

With continuous strategy spaces, space of mixed strategies infinite dimensional!

We will prove this theorem in the next lecture.
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