Outline

- Mechanism design
- Revelation principle
  - Incentive compatibility
  - Individual rationality
- “Optimal” mechanisms

Reading:
- Krishna, Chapter 5
Introduction

- In the next 3 lectures, we will study Mechanism Design, which is an area in economics and game theory that has an engineering perspective.

- The goal is to design economic mechanisms or incentives to implement desired objectives (social or individual) in a strategic setting—assuming that the different members of the society each act rationally in a game theoretic sense.

- Mechanism design has important applications in economics (e.g., design of voting procedures, markets, auctions), and more recently finds applications in networked-systems (e.g., Internet interdomain routing, design of sponsored search auctions).
Auction Theory Viewpoint

- We first study the mechanism design problem in an auction theory context, i.e., we are interested in allocating a single indivisible object among agents.
- An auction is one of many ways that a seller can use to sell an object to potential buyers with unknown values.
- In an auction, the object is sold at a price determined by competition among buyers according to rules set by the seller (auction format), but the seller can use other methods.
- The question then is: what is the “best” way to allocate the object?
- Here, we consider the underlying allocation problem by abstracting away from the details of the selling format.
Model

- We assume a seller has a single indivisible object for sale and there are $N$ potential buyers (or bidders) from the set $\mathcal{N} = \{1, \ldots, N\}$.

- Buyers have private values $X_i$ drawn independently from the distribution $F_i$ with associated density function $f_i$ and support $\mathcal{X}_i = [0, w_i]$.
  - Notice that we allow for asymmetries among the buyers, i.e., the distributions of the values need not be the same for all buyers.

- We assume that the value of the object to the seller is 0.

- Let $\mathcal{X} = \prod_{j=1}^{N} \mathcal{X}_j$ denote the product set of buyers’ values and let $\mathcal{X}_{-i} = \prod_{j \neq i} \mathcal{X}_j$.

- We define $f(x)$ to be the joint density of $x = (x_1, \ldots, x_N)$. Since values are independently distributed, we have $f(x) = f_1(x_1) \times \cdots \times f_N(x_N)$. Similarly, we define $f_{-i}(x_{-i})$ to be the joint density of $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. 

Mechanism

- A selling mechanism $(\mathcal{B}, \pi, \mu)$ has the following components:
  - A set of messages (or bids/strategies) $\mathcal{B}_i$ for each buyer $i$,
  - An allocation rule $\pi : \mathcal{B} \rightarrow \Delta$, where $\Delta$ is the set of probability distributions over the set of buyers $\mathcal{N}$,
  - A payment rule $\mu : \mathcal{B} \rightarrow \mathbb{R}^N$.

- An allocation rule specifies, as a function of messages $b = (b_1, \ldots, b_N)$, the probability $\pi_i(b)$ that $i$ will get the object. Similarly, a payment rule specifies the payment $\mu_i(b)$ that $i$ must make.

- Every mechanism defines a game of incomplete information among the buyers.
  - Strategies: $\beta_i : [0, w_i] \rightarrow \mathcal{B}_i$
  - Payoffs: Expected payoff for a given strategy profile and selling mechanism

- A strategy profile $\beta(\cdot)$ is a Bayesian Nash equilibrium of a mechanism if for all $i$ and for all $x_i$, given the strategies $\beta_{-i}$ of other buyers, $\beta_i(x_i)$ maximizes buyer $i$’s expected payoff.
Direct Mechanisms and Revelation Principle

- A mechanism could be very complicated since we made no assumptions on the message sets $\mathcal{B}_i$.
- A special class of mechanisms, referred to as direct mechanisms, are those for which the set of messages is the same as the set of types (or values), i.e., $\mathcal{B}_i = \mathcal{X}_i$ for all $i$.
- These mechanisms are called “direct” since every buyer is asked directly to report a value.
- Formally a direct mechanism $(Q, M)$ consists of the following components:
  - A function $Q : \mathcal{X} \rightarrow \Delta$, where $Q_i(x)$ is the probability that $i$ will get the object,
  - A function $M : \mathcal{X} \rightarrow \mathbb{R}^N$, where $M_i(x)$ is the payment by buyer $i$.
- If it is a Bayesian Nash equilibrium for each buyer to report (or reveal) their type $x_i$ correctly, we say that the direct mechanism has a truthful equilibrium.
- We refer to the pair $(Q(x), M(x))$ as the outcome of the mechanism.
Revelation Principle

• The following key result, referred to as the revelation principle, allows us to restrict our attention to direct mechanisms.

• More specifically, it shows that the outcomes resulting from any equilibrium of any mechanism can be replicated by a truthful equilibrium of some direct mechanism.

Proposition (Revelation Principle)

Given a mechanism \((B, \pi, \mu)\) and an equilibrium \(\beta\) of that mechanism, there exists a direct mechanism \((Q, M)\), in which

(i) it is a Bayesian Nash equilibrium for each buyer to report his value truthfully,

(ii) the outcomes are the same as in equilibrium \(\beta\) of the original mechanism.

Proof: This follows simply by defining the functions \(Q : \mathcal{X} \rightarrow \Delta\) and \(M : \mathcal{X} \rightarrow \ IR^N\) as \(Q(x) = \pi(\beta(x))\), and \(M(x) = \mu(\beta(x))\). Instead of buyers submitting message \(b_i = \beta(x_i)\), the mechanism asks the buyer to report their value and makes sure the outcome is the same as if they had submitted \(\beta_i(x_i)\).
Revelation Principle

The basic idea behind revelation principle is as follows:

- Suppose that in mechanism \((\mathcal{B}, \pi, \mu)\), each agent finds that, when his type is \(x_i\), choosing \(\beta_i(x_i)\) is his best response to others’ strategies.

- Then, if we have a mediator who says “Tell me your type \(x_i\) and I will play \(\beta_i(x_i)\) for you,” each agent will find truth telling to be an optimal strategy given that all other agents tell the truth.

- In other words, a direct mechanism does the “equilibrium calculations” for the buyers automatically.
Incentive Compatibility

- For a given direct mechanism $(Q, M)$, we define

$$q_i(z_i) = \int_{X_{-i}} Q_i(z_i, x_{-i}) f_{-i}(x_{-i}) dx_{-i},$$

to be the probability that $i$ will get the object when he reports his value to be $z_i$ and all other buyers report their values truthfully.

- Similarly, we define

$$m_i(z_i) = \int_{X_{-i}} M_i(z_i, x_{-i}) f_{-i}(x_{-i}) dx_{-i}$$

to be the expected payment of $i$ when his report is $z_i$ and all other buyers tell the truth.

- The expected payoff of buyer $i$ when his true value is $x_i$ and he reports $z_i$, assuming all others tell the truth, can be written as

$$q_i(z_i)x_i - m_i(z_i).$$
**Incentive Compatibility**

**Definition**

We say that the direct revelation mechanism \((Q, M)\) is *incentive compatible (IC)* if

\[
q_i(x_i)x_i - m_i(x_i) \geq q_i(z_i)x_i - m_i(z_i)
\]

for all \(i, x_i, z_i\).

We refer to the left-hand side of this relation as the equilibrium payoff function denoted by \(U_i(x_i)\), i.e.,

\[
U_i(x_i) = \max_{z_i \in X_i} \{ q_i(z_i)x_i - m_i(z_i) \}.
\]

**Properties under IC:**

- Since \(U_i\) is a maximum of a family of affine functions, it follows that \(U_i\) is a convex function.
- Moreover, it can be seen that incentive compatibility is equivalent to having for all \(z_i\) and \(x_i\)

\[
U_i(z_i) \geq U_i(x_i) + q_i(x_i)(z_i - x_i).
\]
This follows by writing for all \( z_i \) and \( x_i \)
\[
q_i(x_i)z_i - m_i(x_i) = q_i(x_i)x_i - m_i(x_i) + q_i(x_i)(z_i - x_i)
\]
\[
= U_i(x_i) + q_i(x_i)(z_i - x_i).
\]

Eq. (1) implies that for all \( x_i \), \( q_i(x_i) \) is a subgradient of the function \( U_i \) at \( x_i \).

Thus at every point that \( U_i \) is differentiable,

\[
U_i'(x_i) = q_i(x_i).
\]

Since \( U_i \) is convex, this implies that \( q_i \) is a nondecreasing function.

Moreover, we have

\[
U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t_i) dt_i. \tag{2}
\]

This shows that, up to an additive constant, the expected payoff to a buyer in an IC direct mechanism \((Q, M)\) depends only on the allocation rule \( Q \).

From the preceding relations, one can also infer that incentive compatibility is equivalent to the function \( q_i \) being nondecreasing.
Revenue Equivalence

The payoff equivalence derived in the previous slide leads to the following general revenue equivalence principle.

Proposition (Revenue Equivalence)

*If the direct mechanism \((Q, M)\) is incentive compatible, then for all \(i\) and \(x_i\), the expected payment is given by*

\[
m_i(x_i) = m_i(0) + q_i(x_i)x_i - \int_0^{x_i} q_i(t_i)dt_i.
\]

*Thus the expected payments in any two IC mechanisms with the same allocation rule are equivalent up to a constant.*
Revenue Equivalence

Remarks:

- Given two BNE of two different auctions such that for each $i$:
  - For all $(x_1, \ldots, x_N)$, probability of $i$ getting the object is the same,
  - They have the same expected payment at 0 value.

These equilibria generate the same expected revenue for the seller.

- This generalizes the result from last time:
  - Revenue equivalence at the symmetric equilibrium of standard auctions (object allocated to buyer with the highest bid).
Individual Rationality (participation constraints)

- A seller cannot force a bidder to participate in an auction which offers him less expected utility than he could get on his own.
- If he did not participate in the auction, the bidder could not get the object, but also would not pay any money, so his payoff would be zero.
- We say that a direct mechanism \((Q, M)\) is individually rational (IR) if for all \(i\) and \(x_i\), the equilibrium expected payoff satisfies \(U_i(x_i) \geq 0\).
- If the mechanism is IC, then from Eq. (2), individual rationality is equivalent to \(U_i(0) \geq 0\).
- Since \(U_i(0) = -m_i(0)\), individual rationality is equivalent to \(m_i(0) \leq 0\).
Optimal Mechanisms

- Our goal is to design the optimal mechanism that maximizes the expected revenue among all mechanisms that are IC and IR.
- Without loss of generality we can focus on direct revelation mechanisms.
- Consider the direct mechanism \((Q, M)\).
- We can write the expected revenue to the seller as:

\[
E[R] = \sum_{i \in N} E[m_i(X_i)], \quad \text{where}
\]

\[
E[m_i(X_i)] = \int_0^{w_i} m_i(x_i) f_i(x_i) \, dx_i
\]

\[
= m_i(0) + \int_0^{w_i} q_i(x_i) x_i f_i(x_i) \, dx_i - \int_0^{w_i} \int_0^{x_i} q_i(t_i) \, dt_i f_i(x_i) \, dx_i
\]

- Changing the order of integration in the third term, we obtain

\[
E[m_i(X_i)] = m_i(0) + \int_0^{w_i} \left( x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) q_i(x_i) f_i(x_i) \, dx_i
\]

\[
= m_i(0) + \int_{\chi^i} \left( x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) Q_i(x) f(x) \, dx.
\]
Optimal Mechanism Design Problem

- The optimal mechanism design problem can be written as
  \[
  \begin{align*}
  \text{maximize} & \quad E[R] \\
  \text{subject to} & \quad IC(\iff q_i\text{ nondecreasing}) + IR(\iff m_i(0) \leq 0)
  \end{align*}
  \]

- We define the virtual valuation of a buyer with value \( x_i \) as
  \[
  \Psi_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}.
  \]

- We say that the design problem is regular when the virtual valuation \( \Psi_i(x_i) \) is strictly increasing in \( x_i \).

- We next show that under this regularity assumption, we can without loss of generality neglect the IC and the IR constraints.

- The seller should choose \( Q \) and \( M \) to maximize
  \[
  \sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{X}} \left( \sum_{i \in \mathcal{N}} \Psi_i(x_i) Q_i(x) \right) f(x) \, dx.
  \]
Optimal Mechanism

The following is an optimal mechanism:

- **Allocation Rule:**
  \[ Q_i(x) > 0 \iff \Psi_i(x_i) = \max_{j \in N} \Psi_j(x_j) \geq 0. \]

- **Payment Rule:**
  \[ M_i(x) = Q_i(x)x_i - \int_0^{x_i} Q_i(z_i, x_{-i}) dz_i. \]

We finally show that this mechanism satisfies IC and IR.

- We have \( M_i(0, x_{-i}) = 0 \) for all \( x_{-i} \) implying that \( m_i(0) = 0 \), and therefore satisfying IR.

- By the regularity assumption, for any \( z_i < x_i \), we have \( \Psi_i(z_i) < \Psi_i(x_i) \).
  This implies that \( Q_i(z_i, x_{-i}) \leq Q_i(x_i, x_{-i}) \) for all \( x_{-i} \), and therefore \( q_i(z_i) \leq q_i(x_i) \), i.e., \( q_i \) is nondecreasing. Hence, IC is also satisfied.
Optimal Mechanism

- The optimal expected revenue is given by
  \[ E[\max\{\Psi_1(x_1), \ldots, \Psi_N(x_N), 0\}] \]
  i.e., it is the expectation of the highest virtual valuation provided it is nonnegative.

- We define
  \[ y_i(x_{-i}) = \inf\{z_i | \Psi_i(z_i) \geq 0, \Psi_i(z_i) \geq \Psi_j(x_j) \text{ for all } j \neq i\} \]
  i.e., it is the smallest value for \( i \) that wins against \( x_{-i} \).

- Using this, we can write
  \[ Q_i(z_i, x_{-i}) = \begin{cases} 1 & \text{if } z_i > y_i(x_{-i}) \\ 0 & \text{if } z_i < y_i(x_{-i}) \end{cases} \]
Optimal Mechanism

We have

$$\int_0^{x_i} Q_i(z_i, x_{-i}) = \begin{cases} x_i - y_i(x_{-i}) & \text{if } x_i > y_i(x_{-i}) \\ 0 & \text{if } x_i < y_i(x_{-i}) \end{cases}$$

implying that

$$M_i(x) = \begin{cases} y_i(x_{-i}) & \text{if } Q_i(x) = 1 \\ 0 & \text{if } Q_i(x) = 0 \end{cases}$$

This implies that:

- Only the winning buyer pays,
- He pays the smallest value that would result in his winning.
Optimal Mechanism – Symmetric Case

- Suppose that distributions of values are identical across buyers, i.e., for all $i$, we have $f_i = f$. This implies that for all $i$, we have $\Psi_i = \Psi$.
- Note that in this case, we have
  \[ y_i(x_{-i}) = \max\{\Psi^{-1}(0), \max_{j \neq i} x_j\}. \]

**Proposition**

*Assume that the design problem is regular and symmetric. Then a second price auction (Vickrey) with reservation price $r^* = \Psi^{-1}(0)$ is an optimal mechanism.*

- Note that, unlike first and second price auctions, the optimal mechanism is not efficient, i.e., object does not necessarily end up with the person who values it most.