Exercise 6.5:

Consider the Markov process illustrated below. The transitions are labelled by the rate $q_{ij}$ at which those transitions occur. The process can be viewed as a single server queue where arrivals become increasingly discouraged as the queue lengthens. The word time-average below refers to the limiting time-average over each sample-path of the process, except for a set of sample paths of probability 0.

Part a) Find the time-average fraction of time $p_i$ spent in each state $i > 0$ in terms of $p_0$ and then solve for $p_0$. Hint: First find an equation relating $p_i$ to $p_{i+1}$ for each $i$. It also may help to recall the power series expansion of $e^x$.

Solution: From equation (6.36) we know:

$$p_i \frac{\lambda}{i+1} = p_{i+1} \mu, \quad \text{for } i \geq 0$$

By iterating over $i$ we get:

$$p_{i+1} = \frac{\lambda}{\mu (i+1)} p_i = \left( \frac{\lambda}{\mu} \right)^{i+1} \frac{1}{(i+1)!} p_0, \quad \text{for } i \geq 0$$

$$1 = \sum_{i=0}^{\infty} p_i$$

$$= p_0 \left[ 1 + \sum_{i=1}^{\infty} \left( \frac{\lambda}{\mu} \right)^i \frac{1}{i!} \right]$$

$$= p_0 e^{\lambda/\mu}$$

Where the last derivation is in fact the Taylor expansion of the function $e^x$. Thus,

$$p_0 = e^{-\lambda/\mu}$$

$$p_i = \left( \frac{\lambda}{\mu} \right)^i \frac{1}{i!} e^{-\lambda/\mu}, \quad \text{for } i \geq 1$$

Part b) Find a closed form solution to $\sum_i p_i v_i$ where $v_i$ is the departure rate from state $i$. Show that the process is positive recurrent for all choices of $\lambda > 0$ and $\mu > 0$ and explain
intuitively why this must be so.

**Solution:** We observe that $v_0 = \lambda$ and $v_i = \mu + \lambda/(i + 1)$ for $i \geq 1$.

$$\sum_{i \geq 0} p_i v_i = \lambda e^{-\lambda/\mu} + \sum_{i \geq 1} \left\{ (\mu + \lambda/(i + 1)) \left( \frac{\lambda}{\mu} \right)^i \frac{1}{i!} e^{-\lambda/\mu} \right\}$$

$$= \lambda e^{-\lambda/\mu} + \mu e^{-\lambda/\mu} \sum_{i \geq 1} \left( \frac{\lambda}{\mu} \right)^i \frac{1}{i!} + \mu e^{-\lambda/\mu} \sum_{i \geq 1} \left( \frac{\lambda}{\mu} \right)^{i+1} \frac{1}{(i+1)!}$$

$$= \lambda e^{-\lambda/\mu} + \mu e^{-\lambda/\mu} \left( e^{\lambda/\mu} - 1 \right) + \mu e^{-\lambda/\mu} \left( e^{\lambda/\mu} - 1 - \frac{\lambda}{\mu} \right)$$

$$= 2 \mu \left( 1 - e^{-\lambda/\mu} \right)$$

The value of $\sum_i p_i v_i$ is finite for all values of $\lambda > 0$ and $\mu > 0$. So this process is positive recurrent for all choices of transition rates $\lambda$ and $\mu$.

We saw that $P_{i+1} = \frac{\lambda P_i}{\mu(i+1)}$, so $p_i$ must decrease rapidly in $i$ for sufficiently large $i$. Thus the fraction of time spent in very high numbered states must be negligible. This suggests that the steady-state equations for the $p_i$ must have a solution. Since $v_i$ is bounded between $\mu$ and $\mu + \lambda$ for all $i$, it is intuitively clear that $\sum_i v_i p_i$ is finite, so the embedded chain must be positive recurrent.

**Part c)** For the embedded Markov chain corresponding to this process, find the steady state probabilities $\pi_i$ for each $i \geq 0$ and the transition probabilities $P_{ij}$ for each $i, j$.

**Solution:** Since as shown in part (b), $\sum_{i \geq 0} p_i v_i = 1/\left( \sum_{i \geq 0} \pi_i / v_i \right) < \infty$, we know that for all $j \geq 0$, $\pi_j$ is proportional to $p_j v_j$:

$$\pi_j = \frac{P_j v_j}{\sum_{k \geq 0} p_k v_k}, \quad \text{for } j \geq 0$$

So $\pi_0 = \frac{\lambda e^{-\lambda/\mu}}{2\mu(1-e^{-\lambda/\mu})} = \frac{\rho}{2 \rho e^\rho - 1}$ and for $j \geq 1$:

$$\pi_j = \frac{[\mu + \lambda/(j + 1)] \rho j! e^{-\rho}}{2\mu \left( 1 - e^{-\rho} \right) j!}$$

$$= \frac{\mu [1 + \rho/(j + 1)] \rho j e^{-\rho}}{2\mu (1 - e^{-\rho}) j!}$$

$$= \frac{\rho^j}{j!} \left( 1 + \frac{\rho}{j + 1} \right) \frac{1}{2(e^\rho - 1)}, \quad \text{for } j \geq 1$$
The embedded Markov chain will look like:

![Embedded Markov Chain Diagram]

The transition probabilities are:

\[
P_{01} = 1
\]

\[
P_{i,i-1} = \frac{(i + 1)\mu}{\lambda + (i + 1)\mu}, \quad \text{for } i \geq 1
\]

\[
P_{i,i+1} = \frac{\lambda}{\lambda + (i + 1)\mu}, \quad \text{for } i \geq 1
\]

Finding the steady state distribution of this Markov chain gives the same result as found above.

**Part d)** For each \(i\), find both the time-average interval and the time-average number of overall state transitions between successive visits to \(i\).

**Solution:** Looking at this process as a delayed renewal reward process where each entry to state \(i\) is a renewal and the inter-renewal intervals are independent. The reward is equal to 1 whenever the process is in state \(i\).

Given that transition \(n - 1\) of the embedded chain enters state \(i\), the interval \(U_n\) is exponential with rate \(v_i\), so \(E[U_n | X_{n-1} = i] = 1/v_i\). During this \(U_n\) time, reward is 1 and then it is zero until the next renewal of the process.

The total average fraction of time spent in state \(i\) is \(p_i\) with high probability.

So in the steady state, the total fraction of time spent in state \(i\) (\(p_i\)) should be equal to the fraction of time spent in state \(i\) in one inter-renewal interval. The expected length of time spent in state \(i\) in one inter-renewal interval is \(1/v_i\) and the expected inter-renewal interval (\(W_i\)) is what we want to know:

\[
p_i = \frac{U}{W_i} = \frac{1}{v_i W_i}
\]

Thus \(W_i = \frac{1}{p_i v_i}\).

\[
W_0 = \frac{e^\rho}{\lambda}
\]

\[
W_i = \frac{(i + 1)!}{\mu \rho^i (i + 1 + \rho)} e^\rho, \quad \text{for } i \geq 1
\]
Applying Theorem 5.1.4 to the embedded chain, the expected number of transitions, $E[T_{ii}]$ from one visit to state $i$ to the next, is $T_{ii} = 1/\pi_i$.

**Exercise 6.9:**
Let $q_{i,i+1} = 2^{i-1}$ for all $i \geq 0$ and let $q_{i,i-1} = 2^{i-1}$ for all $i \geq 1$. All other transition rates are 0.

**Part a)** Solve the steady-state equations and show that $p_i = 2^{-i-1}$ for all $i \geq 0$.

**Solution:** The defined Markov process can be shown as:

For each $i \geq 0$, $p_i q_{i,i+1} = p_{i+1} q_{i+1,i}$.

$$p_i = \frac{1}{2} p_{i-1}, \quad \text{for } i \geq 1$$

$$1 = \sum_{j \geq 0} p_j = p_0 \left( 1 + \frac{1}{2} + \frac{1}{4} + \cdots \right)$$

So $p_0 = \frac{1}{2}$ and

$$p_i = \frac{1}{2^{i+1}}, \quad i \geq 0$$

**Part b)** Find the transition probabilities for the embedded Markov chain and show that the chain is null-recurrent.

**Solution:**

The embedded Markov chain is:

The steady state probabilities satisfy $\pi_0 = 1/2\pi_1$ and $1/2\pi_i = 1/2\pi_{i+1}$ for $i \geq 1$. So $2\pi_0 = \pi_1 = \pi_2 = \pi_3 = \cdots$. This is a null-recurrent chain, as essentially shown in Exercise 5.2.

**Part c)** For any state $i$, consider the renewal process for which the Markov process starts in state $i$ and renewals occur on each transition to state $i$. Show that, for each $i \geq 1$, the expected inter-renewal interval is equal to 2. Hint: Use renewal reward theory.
Solution:

As explained in Exercise 6.5 part (d), the expected inter-renewal intervals of recurrence of state $i$ ($\overline{W_i}$) satisfies the equation $p_i = \frac{1}{v_i}$. Hence,

$$\overline{W_i} = \frac{1}{v_i p_i} = \frac{1}{2^{-(i+1)/2^i}} = 2.$$  

Where $v_i = q_{i,i+1} + q_{i,i-1} = 2^i$

Part d) Show that the expected number of transitions between each entry into state $i$ is infinite. Explain why this does not mean that an infinite number of transitions can occur in a finite time.

Solution: We have seen in part b) that the embedded chain is null-recurrent. This means that, given $X_0 = i$, for any given $i$, that a return to $i$ must happen in a finite number of transitions (i.e., $\lim_{n \to \infty} F_{ii}(n) = 1$). We have seen many rv’s that have an infinite expectation, but, being rv’s, have a finite sample value WP1.

Exercise 6.14:

A small bookie shop has room for at most two customers. Potential customers arrive at a Poisson rate of 10 customers per hour; They enter if there is room and are turned away, never to return, otherwise. The bookie serves the admitted customers in order, requiring an exponentially distributed time of mean 4 minutes per customer.

Part a) Find the steady state distribution of the number of customers in the shop.

Solution: The arrival rate of the customers is 10 customers per hour and the service time is exponentially distributed with rate 15 customers per hour (or equivalently with mean 4 minutes per customer). The Markov process corresponding to this bookie store is:

$$\begin{align*}
\begin{array}{c}
0 \quad 10 \\
10 \quad 15
\end{array}
\begin{array}{c}
1 \quad 10 \\
15 \quad 2
\end{array}
\end{align*}$$

To find the steady state distribution of this process we use the fact that $p_0 q_{0,1} = p_1 q_{1,0}$ and $p_1 q_{1,2} = p_2 q_{2,1}$. So:

$$
\begin{align*}
10p_0 &= 15p_1 \\
10p_1 &= 15p_2 \\
1 &= p_0 + p_1 + p_2
\end{align*}
$$

Thus, $p_1 = \frac{6}{19}$, $p_0 = \frac{9}{19}$, $p_2 = \frac{4}{19}$. 


Part b) Find the rate at which potential customers are turned away.

Solution:
The customers are turned away when the process is in state 2 and when the process is in state 2, at rate \( \lambda = 10 \) the customers are turned away. So the overall rate at which the customers are turned away is \( \lambda p_2 = \frac{40}{19} \).

Part c) Suppose the bookie hires an assistant; the bookie and assistant, working together, now serve each customer in an exponentially distributed time of mean 2 minutes, but there is only room for one customer (i.e., the customer being served) in the shop. Find the new rate at which customers are turned away.

Solution:
The new Markov process will look like:

\[
\begin{array}{c}
0 \quad 10 \quad 1 \\
30
\end{array}
\]

The new steady state probabilities satisfy 10\(p_0 = 30p_1\) and \(p_0 + p_1 = 1\). Thus, \(p_1 = \frac{1}{4}\) and \(p_0 = \frac{3}{4}\). The customers are turned away if the process is in state 1 and then this happens with rate \(\lambda = 10\). Thus, the overall rate at which the customers are turned away is \(\lambda p_1 = \frac{5}{2}\).

Exercise 6.16:

Consider the job sharing computer system illustrated below. Incoming jobs arrive from the left in a Poisson stream. Each job, independently of other jobs, requires pre-processing in system 1 with probability \(Q\). Jobs in system 1 are served FCFS and the service times for successive jobs entering system 1 are IID with an exponential distribution of mean \(1/\mu_1\). The jobs entering system 2 are also served FCFS and successive service times are IID with an exponential distribution of mean \(1/\mu_2\). The service times in the two systems are independent of each other and of the arrival times. Assume that \(\mu_1 > \lambda Q\) and that \(\mu_2 > \lambda\). Assume that the combined system is in steady state.

Part a) Is the input to system 1 Poisson? Explain.

Solution: Yes. The incoming jobs from the left are Poisson process. This process is split in two processes independently where each job needs a preprocessing in system 1
with probability $Q$. We know that if a Poisson process is split into two processes, each of the processes are also Poisson. So the jobs entering the system 1 is Poisson with rate $\lambda Q$.

**Part b)** Are each of the two input processes coming into system 2 Poisson?

**Solution:** By Burke’s theorem, the output process of a M/M/1 queue is a Poisson process that has the same rate as the input process. So both sequences entering system 2 are Poisson, the first one has rate $Q\lambda$ and the second one has rate $(1 - Q)\lambda$. The overall input is merged process of these two that is going to be a Poisson with rate $\lambda$ (Since these processes are independent of each other.)

**Part d)** Give the joint steady-state PMF of the number of jobs in the two systems. Explain briefly.

**Solution:** We call the number of customers being served in system 1 at time $t$ as $X_1(t)$ and number of customers being served in system 2 at time $t$, as $X_2(t)$.

The splitting of the input arrivals from the left is going to make two independent processes with rates $Q\lambda$ and $(1 - Q)\lambda$. The first process goes into system 1 and defines $X_1(t)$. The output jobs of system 1 at time $t$ is independent of its previous arrivals. Thus the input sequence of system 2 is independent of system 1. The two input processes of system 2 are also independent.

Thus, $X_1(t)$ is the number of customers in an M/M/1 queue with input rate $Q\lambda$ and service rate $\mu_1$ and $X_2(t)$ is the number of customers in an M/M/1 queue with input rate $\lambda$ and service rate $\mu_2$. The total number of the customers in the system is $X(t) = X_1(t) + X_2(t)$ where $X_1(t)$ is independent of $X_2(t)$.

System one can be modeled as a birth death process where $q_{i,i+1} = Q\lambda$ and $q_{i,i-1} = \mu_1$. Thus, $\Pr\{X_1 = i\} = (1 - \rho_1)\rho_1^i$ in the steady state where $\rho_1 = Q\lambda/\mu_1$. The same is true for system 2, thus, $\Pr\{X_2 = j\} = (1 - \rho_2)\rho_2^j$ in the steady state where $\rho_2 = \lambda/\mu_2$. 

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Due to the independency of $X_1$ and $X_2$,
\[
\Pr\{X = k\} = \Pr\{X_1 + X_2 = k\} = \sum_{i=0}^{k} \Pr\{X_1 = i, X_2 = k-i\} = \sum_{i=0}^{k} \Pr\{X_1 = i\} \Pr\{X_2 = k-i\} = \sum_{i=0}^{k} (1 - \rho_1)\rho_1^i (1 - \rho_2)\rho_2^{k-i} = (1 - \rho_1)(1 - \rho_2)\rho_2^k \sum_{i=0}^{k} (\rho_1/\rho_2)^i = (1 - \rho_1)(1 - \rho_2)\rho_2^k - \rho_1^k/\rho_2 - \rho_1.
\]

**Part e)** What is the probability that the first job to leave system 1 after time $t$ is the same as the first job that entered the entire system after time $t$?

**Solution:**
The first job that enters the system after time $t$ is the same as the first job to leave system 1 after time $t$ if and only if $X_1(t) = 0$ (system 1 should be empty at time $t$, unless other jobs will leave system 1 before the specified job) and the first entering job to the whole system needs preprocessing and is routed to system 1 (and should need) which happens with probability $Q$. Since these two events are independent, the probability of the desired event will be $(1 - \rho_1)Q = (1 - \frac{Q}{\mu_1})Q$.

**Part f)** What is the probability that the first job to leave system 2 after time $t$ both passed through system 1 and arrived at system 1 after time $t$?

**Solution:** This is the event that both systems are empty at time $t$ and the first arriving job is routed to system 1 and is finished serving in system 1 before the first job without preprocessing enters system 2. These three events are independent of each other.

The probability that both systems are empty at time $t$ in steady state is $\Pr\{X_1(t) = 0, X_2(t) = 0\} = (1 - \rho_1)(1 - \rho_2)$.

The probability that the first job is routed to system 1 is $Q$.

The service time of the first job in system 1 is called $Y_1$ which is exponentially distributed with rate $\mu_1$ and the probability that the first job is finished before the first job without preprocessing enters system 2 is $\Pr\{Y_1 < Z\}$ where $Z$ is the r.v. which is the arrival time of the first job that does not need preprocessing. It is also exponentially distributed with rate $(1 - Q)\lambda$. Thus, $\Pr\{Y_1 < Z\} = \frac{\mu_1}{\mu_1 + (1 - Q)\lambda}$. 

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Hence, the total probability of the described event is $(1 - \frac{Q\lambda}{\mu_1})(1 - \frac{\lambda}{\mu_2})Q\frac{\mu_1}{\mu_1 + (1-Q)\lambda}$. 