Solutions to practice problem set 12

**Note:** There is a minor error in the statement of Exercise 7.21, part b. The last equation of that part should be \( Z_j = -(2^n/n^2 - n) \). The error is corrected in the statement here.

**Exercise 7.6** Consider a binary hypothesis testing problem where \( H \) is 0 or 1 and a one dimensional observation \( Y \) is given by \( Y = H + U \) where \( U \) is uniformly distributed over \([-1, 1]\) and is independent of \( H \).

a) Find \( f_{Y|H}(y \mid 0) \), \( f_{Y|H}(y \mid 1) \) and the likelihood ratio \( \Lambda(y) \).

**Solution:** Note that \( f_{Y|H} \) is simply the density of \( U \) shifted by \( H \), i.e.,

\[
f_{Y|H}(y \mid 0) = \begin{cases} 1/2; & -1 \leq y \leq 1 \\ 0; & \text{elsewhere} \end{cases} \quad f_{Y|H}(y \mid 1) = \begin{cases} 1/2; & 0 \leq y \leq 2 \\ 0; & \text{elsewhere} \end{cases}
\]

The likelihood ratio \( \Lambda(y) \) is defined only for \(-1 \leq y \leq 2\) since neither conditional density is nonzero outside this range.

\[
\Lambda(y) = \frac{f_{Y|H}(y \mid 0)}{f_{Y|H}(y \mid 1)} = \begin{cases} \infty; & -1 \leq y < 0 \\ 1; & 0 \leq y \leq 1 \\ 0; & 1 < y \leq 2 \end{cases}
\]

b) Find the threshold test at \( \eta \) for each \( \eta \), \( 0 < \eta < \infty \) and evaluate the conditional error probabilities, \( q_0(\eta) \) and \( q_1(\eta) \).

**Solution:** Since \( \Lambda(y) \) has finitely many (3) possible values, all values of \( \eta \) between any adjacent pair lead to the same threshold test. Thus, for \( \eta > 1 \), \( \Lambda(y) > \eta \), leads to the decision \( \hat{h} = 0 \) if and only if (iff) \( \Lambda(y) = \infty \), i.e., iff \(-1 \leq y < 0\). For \( \eta = 1 \), the rule is the same, \( \Lambda(y) > \eta \) iff \( \Lambda(y) = \infty \), but here there is a ‘don’t care’ case \( \Lambda(y) = 1 \) where \( 0 \leq y \leq 1 \) leads to \( \hat{h} = 1 \) simply because of the convention for the equal case taken in (7.14). Finally for \( \eta < 1 \), \( \Lambda(Y) > \eta \) iff \(-1 \leq y \leq 1\).

Consider \( q_0(\eta) \) (the error probability conditional on \( H = 0 \) when a threshold \( \eta \) is used) for \( \eta > 1 \). Then \( \hat{h} = 0 \) iff \(-1 \leq y < 0\), and thus an error occurs (for \( H = 0 \)) iff \( y \geq 0 \). Thus \( q_0(\eta) = \Pr\{Y \geq 0 \mid H = 0\} = 1/2 \). An error occurs given \( H = 1 \) (still assuming \( \eta > 1 \)) iff \(-1 \leq y < 0\). These values of \( y \) are impossible under \( H = 1 \) so \( q_1(\eta) = 0 \). These error probabilities are the same if \( \eta = 1 \) because of the handling of the don’t care cases.

For \( \eta < 1 \), \( \hat{h} = 0 \) iff \( y \leq 1 \). Thus \( q_0(\eta) = \Pr\{Y > 1 \mid H = 0\} = 0 \). Also \( q_1(\eta) = \Pr\{Y \leq 1 \mid H = 1\} = 1/2 \).

c) Find the error curve \( u(\alpha) \) and explain carefully how \( u(0) \) and \( u(1/2) \) are found (hint: \( u(0) = 1/2 \)).
Solution: We have seen that each $\eta \geq 1$ maps into the pair of error probabilities $(q_0(\eta), q_1(\eta)) = (1/2, 0)$. Similarly, each $\eta < 1$ maps into the pair of error probabilities $(q_0(\eta), q_1(\eta)) = (0, 1/2)$. The error curve contains these points and also contains the straight lines joining these points as shown below (see Figure 7.5). The point $u(0)$ is the value of $q_0(\eta)$ for which $q_1(\eta) = \alpha$. Since $q_1(\eta) = 0$ for $\eta \geq 1$, $q_0(\eta) = 1/2$ for those values of $\eta$ and thus $u(0) = 1/2$. Similarly, $u(1/2) = 0$.

\[
\begin{align*}
(0, 1) & \quad 1/2 \\
q_1(\eta) & \quad q_0(\eta) \\
1 & \quad (1, 0)
\end{align*}
\]

d) Describe a decision rule for which the error probability under each hypothesis is $1/4$. You need not use a randomized rule, but you need to handle the don’t-care cases under the threshold test carefully.

Solution: The don’t care cases arise for $0 \leq y \leq 1$ when $\eta = 1$. With the decision rule of (7.14), these don’t care cases result in $\tilde{h} = 1$. If half of those don’t care cases are decided as $\tilde{h} = 0$, then the error probability given $H = 1$ is increased to $1/4$ and that for $H = 0$ is decreased to $1/4$. This could be done by random choice, or just as easily, by mapping $y > 1/2$ into $\tilde{h} = 1$ and $y \leq 1/2$ into $\tilde{h} = 0$.

Exercise 7.12 a) Use Wald’s equality to show that if $\overline{X} = 0$, then $E[S_J] = 0$ where $J$ is the time of threshold crossing with one threshold at $\alpha > 0$ and another at $\beta < 0$.

Solution: Wald’s equality holds since $E[|J|] < \infty$, which follows from Lemma 7.5.1. Thus $E[S_J] = \overline{X}E[J]$. Since $\overline{X} = 0$, it follows that $E[S_J] = 0$.

b) Obtain an expression for $\Pr\{S_J \geq \alpha\}$. Your expression should involve the expected value of $S_J$ conditional on crossing the individual thresholds (you need not try to calculate these expected values).

Solution: Writing out $E[S_J] = 0$ in terms of conditional expectations,

\[
E[S_J] = \Pr\{S_J \geq \alpha\} E[S_J | S_J \geq \alpha] + \Pr\{S_J \leq \beta\} E[S_J | S_J \leq \beta]
\]

\[
= \Pr\{S_J \geq \alpha\} E[S_J | S_J \geq \alpha] + [1 - \Pr\{S_J \geq \alpha\}]E[S_J | S_J \leq \beta]
\]

Using $E[S_J] = 0$, we can solve this for $\Pr\{S_J \geq \alpha\}$,

\[
\Pr\{S_J \geq \alpha\} = \frac{E[-S_J | S_J \leq \beta]}{E[-S_J | S_J \leq \beta] + E[S_J | S_J \geq \alpha]}.
\]

c) Evaluate your expression for the case of a simple random walk.

Solution: A simple random walk moves up or down only by unit steps, Thus if $\alpha$ and $\beta$ are integers, there can be no overshoot when a threshold is crossed. Thus $E[S_J | S_J \geq \alpha] = \alpha$ and $E[S_J | S_J \leq \beta] = \beta$. Thus $\Pr\{S_J \geq \alpha\} = \frac{\beta}{|\beta| + \alpha}$. If $\alpha$ is non-integer, then a positive
threshold crossing occurs at $[\alpha]$ and a lower threshold crossing at $|\beta|$. Thus, in this
general case $\Pr\{S_J \geq \alpha\} = \frac{|\beta|}{(|\beta| + |\alpha|)}$.

d) Evaluate your expression when $X$ has an exponential density, $f_X(x) = a_1e^{-\lambda x}$ for $x \geq 0$
and $f_X(x) = a_2e^{\mu x}$ for $x < 0$ and where $a_1$ and $a_2$ are chosen so that $\bar{X} = 0$.

**Solution:** Let us condition on $J = n$, $S_n \geq \alpha$, and $S_{n-1} = s$, for $s < \alpha$. The overshoot,
$V = S_J - \alpha$ is then $V = X_n + s - \alpha$. Because of the memoryless property of the exponential,
the density of $V$, conditioned as above, is exponential and $f_V(v) = \lambda e^{-\lambda v}$ for $v \geq 0$. This
does not depend on $n$ or $s$, and is thus the overshoot density conditioned only on $S_J \geq \alpha$.
Thus $\Pr\{S_J \geq \alpha\} = \alpha + 1/\lambda$. In the same way, $\Pr\{S_J \geq \beta\} = \beta - 1/\mu$. Thus

$$\Pr\{S_J \geq \alpha\} = \frac{|\beta| + \mu^{-1}}{\alpha + \lambda^{-1} + |\beta| + \mu^{-1}}$$

Note that it is not necessary to calculate $a_1$ or $a_2$.

**Exercise 7.17** Suppose $\{Z_n; n \geq 1\}$ is a martingale. Show that

$$\mathbb{E}\left[Z_m \mid Z_{n_1}, Z_{n_1-1}, \ldots, Z_{n_1}\right] = Z_{n_1} \text{ for all } 0 < n_1 < n_2 < \ldots < n_i < m.$$  

**Solution:** First observe from Lemma 7.6.1 that

$$\mathbb{E}\left[Z_m \mid Z_{n_1}, Z_{n_1-1}, Z_{n_1-2}, Z_1\right] = Z_{n_1}$$

This is valid for every sample value of every conditioning variable. Thus consider $Z_{n_1-1}$
for example. Since this equation has the same value for each sample value of $Z_{n_1-1}$,
we could take the expected value of this conditional expectation over $Z_{n_1-1}$, getting
$\mathbb{E}[Z_m \mid Z_{n_1}, Z_{n_1-2}, Z_1] = Z_{n_1}$. In the same way, any subset of these conditioning rv’s
could be removed, leaving us with the desired form.

**Exercise 7.21:** a) This exercise shows why the condition $\mathbb{E}\{\mid Z_n \mid \} < \infty$ is required in
Lemma 7.8.1. Let $Z_1 = -2$ and, for $n \geq 1$, let $Z_{n+1} = Z_n[1 + X_n(3n + 1)/(n + 1)]$ where
$X_1, X_2, \ldots$ are IID and take on the values $+1$ and $-1$ with probability $1/2$ each. Show that
$\{Z_n; n \geq 1\}$ is a martingale.

**Solution:** From the definition of $Z_n$ above,

$$\mathbb{E}\left[Z_n \mid Z_{n-1}, Z_{n-2}, \ldots, Z_1\right] = \mathbb{E}\left[Z_{n-1}[1 + X_{n-1}(3n - 2)/n] \mid Z_{n-1}, \ldots, Z_1\right]$$

Since the $X_n$ are zero mean and IID, this is just $\mathbb{E}[Z_{n-1} \mid Z_{n-1}, \ldots, Z_1]$, which is $Z_{n-1}$.
Thus $\{Z_n; n \geq 1\}$ is a martingale.

b) Consider the stopping trial $J$ such that $J$ is the smallest value of $n > 1$ for which
$Z_n$ and $Z_{n-1}$ have the same sign. Show that, conditional on $n < J$, $Z_n = (-2)^n/n$ and,
conditional on $n = J$, $Z_J = -(n - 2)/n^2$.

**Solution:** It can be seen from the iterative definition of $Z_n$ that $Z_n$ and $Z_{n-1}$ have
different signs if $X_{n-1} = -1$ and have the same sign if $X_{n-1} = 1$. Thus the stopping
trial is the smallest trial \( n \geq 2 \) for which \( X_{n-1} = 1 \). Thus for \( n < J \), we must have \( X_i = -1 \) for \( 1 \leq i < n \). For \( n = 2 < J \), \( X_1 \) must be \(-1\), so from the formula above, 
\[ Z_2 = Z_1[1 - 4/2] = 2. \] Thus \( Z_n = (-2)^n/n \) for \( n = 2 < J \). We can use induction now for arbitrary \( n < J \). Thus for \( X_n = -1 \),
\[ Z_{n+1} = Z_n \left[ 1 - \frac{3n + 1}{n + 1} \right] = \frac{(-2)^n}{n} \cdot \frac{-2n}{n + 1} = \frac{(-2)^{n+1}}{n + 1} \]

The remaining task is to compute \( Z_n \) for \( n = J \). Using the result just derived for \( n = J-1 \) and using \( X_{J-1} = 1 \),
\[ Z_J = Z_{J-1} \left[ 1 + \frac{3(J-1) + 1}{J} \right] = \frac{(-2)^{J-1}}{J} \cdot \frac{4J - 2}{J} = \frac{(-2)^J(2J - 1)}{J(J - 1)} \]

c) Show that \( \mathbb{E} |Z_J| \) is infinite, so that \( \mathbb{E}[Z_J] \) does not exist according to the definition of expectation, and show that \( \lim_{n \to \infty} \mathbb{E}[Z_n \mid J > n] \mathbb{P}(J > n) = 0 \).

**Solution:** We have seen that \( J = n \) if and only if \( X_i = -1 \) for \( 1 \leq i \leq n - 2 \) and \( X_{n-1} = 1 \). Thus \( \mathbb{P}(J = n) = 2^{-n+1} \) so
\[ \mathbb{E} |Z_J| = \sum_{n=2}^{\infty} 2^{n-1} \cdot \frac{2^n(2n-1)}{n(n-1)} = \sum_{n=2}^{\infty} \frac{2(2n-1)}{n(n-1)} \geq \sum_{n=2}^{\infty} \frac{4}{n} = \infty, \]

since the harmonic series diverges.

Finally, we see that \( \mathbb{P}(J > n) = 2^{n-1} \) since this event occurs if and only if \( X_i = -1 \) for \( 1 \leq i < n \). Thus
\[ \mathbb{E}[Z_n \mid J > n] \mathbb{P}(J > n) = \frac{2^{-n+1}2^n}{n} = 2/n \to 0 \]

Section 7.8 explains the significance of this exercise.

**Exercise 7.29** Let \( \{Z_n; n \geq 1\} \) be a martingale, and for some integer \( m \), let \( Y_n = Z_{n+m} - Z_m \).

a) Show that \( \mathbb{E}[Y_n \mid Z_{n+m-1}, Z_{n+m-2}, \ldots, Z_m = z_m, \ldots, Z_1 = z_1] = z_{n+m-1} - z_m \).

**Solution:** This is more straightforward if the desired result is written in the more abbreviated form
\[ \mathbb{E}[Y_n \mid Z_{n+m-1}, Z_{n+m-2}, \ldots, Z_m, \ldots, Z_1] = Z_{n+m-1} - Z_m. \]

Since \( Y_n = Z_{n+m} - Z_m \), the left side above is
\[ \mathbb{E}[Z_{n+m} - Z_m \mid Z_{n+m-1}, \ldots, Z_1] = Z_{n+m-1} - \mathbb{E}[Z_m \mid Z_{n+m-1}, \ldots, Z_m, \ldots, Z_1] \]

Given sample values for each conditioning rv on the right of the above expression, and particularly given that \( Z_m = z_m \), the expected value of \( Z_m \) is simply the conditioning
value $z_m$ for $Z_m$. This is one of those strange things that are completely obvious, and yet somehow obscure. We then have $E[Y_n \mid Z_{n+m-1}, \ldots, Z_1] = Z_{n+m-1} - Z_m$.

b) Show that $E[Y_n \mid Y_{n-1} = y_{n-1}, \ldots, Y_1 = y_1] = y_{n-1}$.

**Solution:** In abbreviated form, we want to show that $E[Y_n \mid Y_{n-1}, \ldots, Y_1] = Y_{n-1}$. We showed in part a) that $E[Y_n \mid Z_{n+m-1}, \ldots, Z_1] = Y_{n-1}$. For each sample point $\omega$, $Y_{n-1}(\omega), \ldots, Y_1(\omega)$ is a function of $Z_{n+m-1}(\omega), \ldots, Z_1(\omega)$. Thus, the rv $E[Y_n \mid Z_{n+m-1}, \ldots, Z_1]$ specifies the rv $E[Y_n \mid Y_{n-1}, \ldots, Y_1]$, which then must be $Y_{n-1}$.

c) Show that $E[|Y_n|] < \infty$. Note that b) and c) show that $\{Y_n; n \geq 1\}$ is a martingale.

**Solution:** Since $Y_n = Z_{n+m} - Z_m$, we have $|Y_n| \leq |Z_{n+m}| + |Z_m|$. Since $\{Z_n; n \geq 1\}$ is a martingale, $E[|Z_n|] < \infty$ for each $n$ so

$$E[|Y_n|] \leq E[|Z_{n+m}|] + E[|Z_m|] < \infty.$$