Lecture 16: Renewals and Countable-state Markov

Outline:

• Review major renewal theorems
• Age and duration at given $t$
• Countable-state Markov chains

Sample-path time average (strong law for renewals)

$$\Pr\left\{ \lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{X} \right\} = 1.$$  

Ensemble & time average (elementary renewal thm)

$$\lim_{t \to \infty} \mathbb{E}\left[ \frac{N(t)}{t} \right] = \frac{1}{X}.$$  

Ensemble average (Blackwell’s thm); $m(t) = \mathbb{E}[N(t)]$

$$\lim_{t \to \infty} \left[ m(t+\lambda) - m(t) \right] = \frac{\lambda}{X} \quad \text{Arith. } X, \text{ span } \lambda$$

$$\lim_{t \to \infty} \left[ m(t+\delta) - m(t) \right] = \frac{\delta}{X} \quad \text{Non-Arith. } X, \text{ any } \delta > 0.$$
\[
\lim_{t \to \infty} [m(t+\lambda) - m(t)] = \frac{\lambda}{X} \quad \text{Arith. } X, \text{ span } \lambda
\]
can be rewritten as
\[
\lim_{n \to \infty} \Pr\{\text{renewal at } n\lambda\} = \frac{\lambda}{X} \quad \text{Arith. } X, \text{ span } \lambda
\]
If we model an arithmetic renewal process as a Markov chain starting in the renewal state 0, this essentially says \( P^n_{00} \to \pi_0 \).
\[
\lim_{t \to \infty} [m(t+\delta) - m(t)] = \frac{\delta}{X} \quad \text{Non-Arith. } X, \text{ any } \delta > 0
\]
This is the best one could hope for. Note that
\[
\lim_{\delta \to 0} \lim_{t \to \infty} \frac{m(t+\delta) - m(t)}{\delta} = \frac{1}{X}
\]
but the order of the limits can’t be interchanged.

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**Age and duration at given \( t \)**

\[ 0 \quad S_1 \quad S_2 \quad S_3 \]

\[ \begin{array}{c}
0 & X_1 & X_2 & Z(t) & \tilde{X}(t) & N(t) \\
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\end{array} \]

\( \tilde{X}(t) = X_{N(t)} \)

Assume an arithmetic renewal process of span 1.

For integer \( t \), \( Z(t) = i \geq 0 \) and \( \tilde{X}(t) = k > i \) iff there are successive arrivals at \( t - i \) and \( t - i + k \).

Let \( q_j = \Pr\{\text{arrival at time } j\} = \sum_{n \geq 1} p_{S_n}(j) \) and let \( q_0 = 1 \) (nominal arrival at time 0). Then
\[
p_{Z(t), \tilde{X}(t)}(i, k) = q_{t-i} p_X(k) \quad \text{for } 0 \leq i \leq t; \ k > i
\]
\[ p_{Z(t),\tilde{X}(t)}(i, k) = q_{t-i}p_X(k) \quad \text{for } 0 \leq i \leq t; \ k > i \]

Note that

\[ q_i = \Pr\{\text{arrival at } j\} = E[\text{arrival at } j] = m(i) - m(i-1), \]

so by Blackwell, \( \lim_{j \to \infty} 1/X. \)

\[
\lim_{t \to \infty} p_{Z(t),\tilde{X}(t)}(i, k) = \frac{p_X(k)}{X} \quad \text{for } k > i \geq 0.
\]

\[
\lim_{t \to \infty} p_Z(t)(i) = \frac{\sum_{k=i+1}^{\infty} p_X(k)}{X} = \frac{F_X^c(i)}{X} \quad \text{for } i \geq 0.
\]

\[
\lim_{t \to \infty} p_{\tilde{X}(t)}(k) = \frac{\sum_{i=0}^{k-1} p_X(k)}{X} = \frac{kp_X(k)}{X} \quad \text{for } k \geq 1.
\]

Now look at asymptotic expected duration:

\[ \lim_{t \to \infty} E[\tilde{X}(t)] = \sum_{k=1}^{\infty} k \cdot kp_X(k)/X = E[X^2]/X \]

This is the same as the sample-path average, but now we can look at the finite \( t \) case. More important, we get a different interpretation.

For a given \( \tilde{X} = k \), there are \( k \) equiprobable choices for age; for each choice, the joint \( Z, \tilde{X} \) PMF is \( p_X(k)/X \). Thus large durations are enhanced relative to inter-renewals.

The expected age (after some work) is \( E[X^2]/2X - \frac{1}{2}. \) This is at integer values of large \( t \). The age increases linearly with slope 1 to the next integer value and then drops by 1.
**Countable -state Markov chains**

The biggest change from finite-state Markov chains to countable-state chains is the concept of a recurrent class. Example:

![Graph of a Markov chain](image)

This Markov chain models a Bernoulli ±1 process. The state at time $n$ is $S_n = X_1 + X_2 + \cdots + X_n$. The state $S_n$ at time $n$ is $j = 2k - n$ where $k$ is the number of positive transitions in the $n$ trials.

All states communicate and have a period $d = 2$; $\sigma^2_{S_n} = n[1 - (p - q)^2]$. $P^n_{0,j}$ approaches 0 at least as $1/\sqrt{n}$ for every $j$.

Another example (called a birth-death chain)

![Graph of a birth-death chain](image)

In this case, if $p > 1/2$, the state drifts to the right and $P^n_{0,j}$ approaches 0 for all $j$. If $p < 1/2$, it drifts to the left and keeps bumping state 0.

A truncated version of this was analyzed in the homework. With $p > 1/2$, the steady-state increases to the right, with $p < 1/2$, it increases to the left, and at $p = 1/2$ it is uniform.

As the truncation point increases, the ‘steady-state’ remains positive only for $p < 1/2$. 

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We want to define recurrent to mean that, given $X_0 = i$, there is a future return to state $i$ WP1. We will see that the birth-death chain above is recurrent if $p < 1/2$ and not recurrent if $p > 1/2$. The case $p = 1/2$ is strange and will be called null-recurrent.

We can use renewal theory to study recurrent chains, but first must understand first-passage-times.

**Def:** The first-passage-time probability, $f_{ij}(n)$, is

$$ f_{ij}(n) = \Pr\{X_n=j, X_{n-1} \neq j, X_{n-2} \neq j, \ldots, X_1 \neq j | X_0 = i\}. $$

It’s the probability, given $X_0 = i$, that $n$ is the first epoch at which $X_n = j$. Then

$$ f_{ij}(n) = \sum_{k \neq j} P_{ik} f_{kj}(n-1); \quad n > 1; \quad f_{ij}(1) = P_{ij}. $$

Recall that Chapman-Kolmogorov says

$$ P^n_{ij} = \sum_k P_{ik} P^{n-1}_{kj}, $$

so the difference between $f_{ij}(n)$ and $P^n_{ij}$ is only in cutting off the outputs from $j$ (as before in finding expected first-passage-times.

Let $F_{ij}(n) = \sum_{m \leq n} f_{ij}(m)$ be the probability of reaching $j$ by time $n$ or before. If $\lim_{n \to \infty} F_{ij}(n) = 1$, there is a rv $T_{ij}$ with distribution function $F_{ij}$ that is the first-passage-time rv.
We can also express $F_{ij}(n)$ as

$$F_{ij}(n) = P_{ij} + \sum_{k \neq j} P_{ik}F_{kj}(n-1); \quad n > 1; \quad F_{ij}(1) = P_{ij}$$

Since $F_{ij}(n)$ is nondecreasing in $n$, the limit $F_{ij}(\infty)$ must exist and satisfy

$$F_{ij}(\infty) = P_{ij} + \sum_{k \neq j} P_{ik}F_{kj}(\infty).$$

Unfortunately, choosing $F_{ij}(\infty) = 1$ for all $i, j$ satisfies these equations. The correct solution turns out to be the smallest set of $F_{ij}(\infty)$ that satisfies these equations.

If $F_{jj}(\infty) = 1$, then an eventual return from state $j$ occurs with probability 1 and the sequence of returns is the sequence of renewal epochs in a renewal process.

If $F_{jj}(\infty) = 1$, then there is a rv $T_{jj}$ with the distribution function $F_{jj}(n)$ and $j$ is recurrent. The renewal process of returns to $j$ then has inter-renewal intervals with the distribution function $F_{jj}(n)$.

From renewal theory, the following are equivalent:

1) state $j$ is recurrent.

2) $\lim_{t \to \infty} N_{jj}(t) = \infty$ with probability 1.

3) $\lim_{t \to \infty} E[N_{jj}(t)] = \infty$.

4) $\lim_{t \to \infty} \sum_{1 \leq n \leq t} P_{jj}^n = \infty$.

None of these imply that $E[T_{jj}] < \infty$. 
Two states are in the same class if they communicate (same as for finite-state chains).

If states $i$ and $j$ are in the same class then either both are recurrent or both transient (not recurrent).

**Pf:** If $j$ is recurrent, then $\sum_n P^n_{jj} = \infty$. Then

$$\sum_{n=1}^{\infty} P^n_{ii} \geq \sum_{k=1}^{\infty} P^m_{ij} P^k_{jj} P^\ell_{jk} = \infty$$

All states in a class are recurrent or all are transient.

By the same kind of argument, if $i, j$ are recurrent, then $F_{ij}(\infty) = 1$.

If a state $j$ is recurrent, then $T_{jj}$ might or might not have a finite expectation.

**Def:** If $E[T_{jj}] < \infty$, $j$ is positive recurrent. If $T_{jj}$ is a rv and $E[T_{jj}] = \infty$, then $j$ is null recurrent. Otherwise $j$ is transient.

For $p = 1/2$, each state in each of the following is null recurrent.