A sequence \{Z_n; n \geq 1\} of rv’s is a martingale if

\[ E[Z_n | Z_{n-1}, Z_{n-2}, \ldots, Z_1] = Z_{n-1}; \quad E[|Z_n|] < \infty \quad (1) \]

for all \( n \geq 1 \).

Recall that \( E[Z_n | Z_{n-1}, \ldots, Z_1] \) is a rv that maps each sample point \( \omega \) to the conditional expectation of \( Z_n \) conditional on \( Z_1(\omega), \ldots, Z_{n-1}(\omega) \). For a martingale, this expectation must be the rv \( Z_{n-1} \).

Lemma: For a martingale, \( \{Z_n; n \geq 1\} \), and for \( n > i \geq 1 \),

\[ E[Z_n | Z_i, Z_{i-1}, \ldots, Z_1] = Z_i; \quad E[Z_n] = E[Z_i] \]

Can you figure out why \( E[Z_n | Z_m, \ldots, Z_1] = Z_n \) for \( m \geq n \)?

Note that \( E[Z_n | Z_1] = Z_1 \) and \( E[Z_n] = E[Z_1] \) for all \( n > 1 \).
Simple Examples of martingales

1) Zero-mean RW: If $Z_n = \sum_{i=1}^{n} X_i$ where $\{X_i; i \geq 1\}$ are IID and zero mean, then $\{Z_n; n \geq 1\}$ is a martingale.

2) If $Z_n = \sum_{i=1}^{n} X_i$ where $\mathbb{E}[X_i | X_{i-1}, \ldots, X_1] = 0$ for each $i \geq 1$, then $\{Z_n; n \geq 1\}$ is a martingale.

3) Let $X_i = U_i Y_i$ where $\{U_i; i \geq 1\}$ are IID, equiprobable $\pm 1$. The $Y_i$ are independent of the $U_i$. Then $\{Z_n; n \geq 1\}$, where $Z_n = X_1 + \cdots + X_n$, is a martingale.

4) Product form martingales: Let $\{X_i; i \geq 1\}$ be a sequence of IID unit-mean r.v.'s. Then $\{Z_n; n \geq 1\}$, where $Z_n = X_1 X_2 \cdots X_n$, is a martingale.

If $p_X(0) = p_X(2) = 1/2$, then $p_{Z_n}(2^n) = 2^{-n}$ and $p_{Z_n}(0) = 1 - 2^{-n}$. Thus, $\lim_{n \to \infty} Z_n = 0$ WP1 and $\lim_{n \to \infty} \mathbb{E}[Z_n] = 1$.

Sub- and supermartingales are sequences $\{Z_n; n \geq 1\}$ with $\mathbb{E}[|Z_n|] < \infty$ for which inequalities replace the equalities of martingales. For all $n \geq 1$,

$$
\mathbb{E}[Z_n | Z_{n-1}, \ldots, Z_1] \geq Z_{n-1} \quad \text{submartingale}
$$

$$
\mathbb{E}[Z_n | Z_{n-1}, \ldots, Z_1] \leq Z_{n-1} \quad \text{supermartingale}
$$

If $\{Z_n; n \geq 1\}$ is a submartingale, then $\{-Z_n; n \geq 1\}$ is a supermartingale and vice-versa, so we consider only submartingales. For submartingales,

$$
\mathbb{E}[Z_n | Z_i, \ldots, Z_1] \geq Z_i \quad \text{for all } n > i > 0
$$

$$
\mathbb{E}[Z_n] \geq \mathbb{E}[Z_i] \quad \text{for all } n > i > 0
$$
Convex functions

A function $h(x), \mathbb{R} \rightarrow \mathbb{R}$, is convex if each tangent to the curve lies on or below the curve. The condition $h''(x) \geq 0$ is sufficient but not necessary.

Lemma (Jensen's inequality): If $h$ is convex and $Z$ is a rv with finite expectation, then

$$h(E[Z]) \leq E[h(Z)]$$

Jensen's inequality leads to the following theorem. See proof in text.

Thm 7.8.1: If $\{Z_n; n \geq 1\}$ is a martingale or submartingale, if $h$ is convex, and if $E[|h(Z_n)|] < \infty$ for all $n$, then $\{h(Z_n); n \geq 1\}$ is a submartingale.

For example, if $\{Z_n; n \geq 1\}$ is a martingale, then $\{|Z_n|; n \geq 1\}$, $\{Z_n^2; n \geq 1\}$ and $\{e^{Z_n}; n \geq 1\}$ are submartingales if the marginal expected values exist.
Stopped martingales

The definition of a stopping trial $J$ for a stochastic process $\{Z_n; n \geq 1\}$ applies to any process. That is, $J$ must be a rv and $\{J = n\}$ must be specified by $\{Z_1, \ldots, Z_n\}$.

A possibly defective rv $J$ is a mapping from $\Omega$ to the extended reals $\mathbb{R}^+$ where $\{J = \infty\}$ and $\{J = -\infty\}$ might have positive probability. The other provisos of rv's still hold.

A possibly defective stopping trial is thus a stopping rule in which stopping may never happen (such as RW's with a single threshold).

A stopped process $\{Z^*_n; n \geq 1\}$ for a possibly defective stopping time $J$ on a process $\{Z_n; n \geq 1\}$ satisfies $Z^*_n = Z_n$ if $n \leq J$ and $Z^*_n = Z_J$ if $n > J$.

For example, a given gambling strategy, where $Z_n$ is the net worth at time $n$, could be modified to stop when $Z_n$ reaches some given value. Then $Z^*_n$ would remain at that value forever after, while $Z_n$ follows the original strategy.

Theorem: If $J$ is a possibly defective stopping rule for a martingale (submartingale), $\{Z_n; n \geq 1\}$, then the stopped process $\{Z^*_n; n \geq 1\}$ is a martingale (submartingale).

Pf: Obvious??? The intuition here is that before stopping occurs, $Z^*_n = Z_n$, so $Z^*_n$ satisfies the martingale (subm.) condition. Afterwards, $Z^*_n$ is constant, so it again satisfies the martingale (subm) condition.
Proof that \( \{ Z_n^*, n \geq 1 \} \) is a martingale: Note that

\[
Z_n^* = \sum_{m=1}^{n-1} Z_m 1_{J=m} + Z_n 1_{J \geq n}
\]

Thus \( |Z_n^*| \leq \sum_{m<n} |Z_m| + |Z_n| \). Thus means that \( E[|Z_n^*|] < \infty \) since it is bounded by the sum of \( n \) finite numbers.

Next, let \( \tilde{Z}^{(n-1)} \) denote \( Z_{n-1}, \ldots, Z_1 \) and consider

\[
E[Z_n^* | \tilde{Z}^{(n-1)}] = \sum_{m<n} E[Z_m 1_{J=m} | \tilde{Z}^{(n-1)}] + E[Z_n 1_{J \geq n} | \tilde{Z}^{(n-1)}]
\]

\[
E[Z_m 1_{J=m} | \tilde{Z}^{(n-1)}] = \begin{cases} 
    z_m; & \text{if } J = m \\
    0; & \text{if } J \neq m.
\end{cases} \text{ for } m < n
\]

\[
E[Z_n 1_{J=m} | \tilde{Z}^{(n-1)}] = Z_n 1_{J=m}.
\]

\[
E[Z_n 1_{J=n} | \tilde{Z}^{(n-1)}] = Z_{n-1} 1_{J \geq n}
\]

\[
E[Z_n^* | \tilde{Z}^{(n-1)}] = \sum_{m<n} Z_m 1_{J=m} + Z_n 1_{J \geq n}
\]

\[
= \sum_{m<n-1} Z_m 1_{J=m} + Z_n 1_{J=n-1} + 1_{J \geq n}
\]

\[
= Z_{n-1}^*
\]

This shows that \( E[Z_n^* | \tilde{Z}^{(n-1)}] = Z_{n-1}^* \). To show that \( \{ Z_n^*, n \geq 1 \} \) is a martingale, though, we must show that

\[
E[Z_n^* | \tilde{Z}^{* (n-1)}] = Z_{n-1}^*.
\]

However, \( \tilde{Z}^{* (n-1)} \) is a function of \( \tilde{Z}^{(n-1)} \).

For every sample point \( \tilde{z}^{(n-1)} \) of \( \tilde{Z}^{(n-1)} \) leading to a given \( \tilde{z}^{* (n-1)} \) of \( \tilde{Z}^{* (n-1)} \), we have

\[
E[Z_n^* | \tilde{Z}^{(n-1)} = \tilde{z}^{(n-1)}] = z_{n-1}^*
\]

and thus

\[
E[Z_n^* | \tilde{Z}^{* (n-1)} = \tilde{z}^{* (n-1)}] = z_{n-1}^*.
\]

QED
A consequence of the theorem, under the same assumptions, is that

\[
E[Z_1] \leq E[Z_n^*] \leq E[Z_n] \quad \text{(submartingale)}
\]
\[
E[Z_1] = E[Z_n^*] = E[Z_n] \quad \text{(martingale)}
\]

This is also almost intuitively obvious and proved in Section 7.8.

Recall the generating function product martingale for a random walk. That is, let \( \{X_n; n \geq 1\} \) be IID and \( \{S_n; n \geq 1\} \) be a random walk where \( S_n = X_1 + \cdots + X_n \).

Then for \( r \) such that \( \gamma(r) \) exists, let \( Z_n = \exp[rS_n - n\gamma(r)] \). Then \( \{Z_n; n \geq 1\} \) is a martingale and \( E[Z_n] = 1 \) for all \( n \geq 1 \).

For \( r \) such that \( \gamma(r) \) exists, let \( Z_n = \exp[rS_n - n\gamma(r)] \). Then \( \{Z_n; n \geq 1\} \) is a martingale and \( E[Z_n] = 1 \) for all \( n \geq 1 \).

Let \( J \) be the nondefective stopping time that stops on crossing either \( \alpha > 0 \) or \( \beta < 0 \). Then \( E[Z_n^*] = 1 \) for all \( n \geq 1 \).

Also, \( \lim_{n \to \infty} Z_n^* = Z_J \) WP1 and

\[
E[Z_J] = E[\exp[rS_J - J\gamma(r)]] = 1
\]

This is Wald’s identity in a more general form. The connection of \( \lim_n Z_n^* \) to \( Z_J \) needs more care (see Section 7.8), but this shows the power of martingales.
Kolmogorov's submartingale inequality

Thm: Let \( \{Z_n; n \geq 1\} \) be a non-negative submartingale. Then for any positive integer \( m \) and any \( a > 0 \),

\[
\Pr \left\{ \max_{1 \leq i \leq m} Z_i \geq a \right\} \leq \frac{E[Z_m]}{a}. \tag{2}
\]

If we replace the max with \( Z_m \), this is the lowly but useful Markov inequality.

Proof: Let \( J \) be the stopping time defined as the smallest \( n \leq m \) such that \( Z_n \geq a \).

If \( Z_n \geq a \) for some \( n \leq m \), then \( J \) is the smallest \( n \) for which \( Z_n \geq a \).

If \( Z_n < a \) for all \( n \leq m \), then \( J = m \). Thus the process must stop by time \( m \), and \( Z_J \geq a \) iff \( Z_n \geq a \) for some \( n \leq m \). Thus

\[
\Pr \left\{ \max_{1 \leq n \leq m} Z_n \geq a \right\} = \Pr\{Z_J \geq a\} \leq \frac{E[Z_J]}{a}.
\]

Since the process must be stopped by time \( m \), we have \( Z_J = Z_m^* \).

\( E[Z_m^*] \leq E[Z_m] \), so the right hand side above is less than or equal to \( E[Z_m]/a \), completing the proof.
The Kolmogorove submartingale inequality is a strengthening of the Markov inequality. The Chebyshev inequality is strengthened in the same way.

Let \( \{Z_n; n \geq 1\} \) be a martingale with \( \mathbf{E}[Z_n^2] < \infty \) for all \( n \geq 1 \). Then

\[
\Pr\left\{ \max_{1 \leq n \leq m} |Z_n| \geq b \right\} \leq \frac{\mathbf{E}[Z_m^2]}{b^2}; \quad \text{for all integer } m \geq 2, \text{all } b > 0.
\]

Let \( \{S_n; n \geq 1\} \) be a RW with \( S_n = X_1 + \cdots + X_n \) where each \( X_i \) has mean \( \bar{X} \) and variance \( \sigma^2 \). Then for any positive integer \( m \) and any \( \epsilon > 0 \),

\[
\Pr\left\{ \max_{1 \leq n \leq m} |S_n - n\bar{X}| \geq m\epsilon \right\} \leq \frac{\sigma^2}{m\epsilon^2}.
\]

**SLLN for IID rv's with a variance**

Thm: Let \( \{X_i; i \geq 1\} \) be a sequence of IID random variables with mean \( \bar{X} \) and standard deviation \( \sigma < \infty \). Let \( S_n = X_1 + \cdots + X_n \). Then for any \( \epsilon > 0 \),

\[
\Pr\left\{ \lim_{n \to \infty} \frac{S_n}{n} = \bar{X} \right\} = 1
\]

Idea of proof:

\[
\Pr\left\{ \bigcup_{m=k}^{\infty} \left\{ \max_{1 \leq n \leq 2^m} |S_n - n\bar{X}| \geq 2^m\epsilon \right\} \right\} \leq \sum_{m=k}^{\infty} \frac{\sigma^2}{2^m\epsilon^2} = \frac{2\sigma^2}{2^k\epsilon^2}
\]

Then lower bound the left term to

\[
\Pr\left\{ \bigcup_{m=k}^{\infty} \left\{ \max_{2^{m-1} \leq n \leq 2^m} |S_n - n\bar{X}| \geq 2n\epsilon \right\} \right\}
\]
The martingale convergence theorem

Thm: Let \( \{Z_n; n \geq 1\} \) be a martingale and assume that there is some finite \( M \) such that \( E[|Z_n|] \leq M \) for all \( n \). Then there is a random variable \( Z \) such that, \( \lim_{n \to \infty} Z_n = Z \) \( \text{WP1} \).

The text proves the theorem with the additional constraint that \( E[Z_n^2] \) is bounded. Either bounded \( E[Z_n^2] \) or bounded \( E[|Z_n|] \) is a very strong constraint, but the theorem is still very powerful.

For a branching process \( \{X_n; n \geq 1\} \) where the number \( Y \) of offspring of an element has \( Y > 1 \), we saw that \( \{X_n/Y^n; n \geq 1\} \) is a martingale satisfying the constraint, so \( X^n/Y^n \to Z \) \( \text{WP1} \).