Chapter 3

Power Factor and Measures of Distortion

Definitions and Identities

Two functions $X$ and $Y$ are orthogonal over $[a, b]$ if:

$$\int_{a}^{b} X(t)Y(t)dt = 0$$  \hspace{1cm} (3.1)

Now:

- $\int_{0}^{2\pi} \sin(m\omega t) \sin(n\omega t + \phi) d\omega t = 0$, if $n \neq m$ $\Rightarrow$ sinusoids of different frequencies are orthogonal.
• \[ \int_0^{2\pi} \sin(\omega t) \cos(\omega t) d\omega t = 0 \Rightarrow \text{sine and cosine are orthogonal.} \]

In general:

\[
\frac{1}{2\pi} \int_0^{2\pi} \sin(\omega t) \sin(\omega t + \phi) = \frac{1}{2} \cos \phi
\]

These definitions will be useful for calculating power, etc.

Suppose we plug a resistor into the wall.

\[ P = \langle Vi \rangle \]
\[ = V_{RMS} i_{RMS} \]
\[ = i^2_{RMS} R \quad (3.3) \]

The fuse is rated for a specific RMS current. Above that, it will blow so that dissipation in \( R_{wire} \) does not start a fire. Neglecting \( R_{wire} \), for 115V_{AC \ RMS}, 15A_{RMS} fuse, we get \( \sim 1.7\text{ kW max from wall.} \)

Suppose instead we plug an inductor into the wall.

Neglecting \( R_{wire} \):
\[
i = -\frac{V_s}{\omega L} \cos(\omega t)
\] (3.4)

\[
< P > = \frac{1}{2\pi} \int V(t) i(t) d(\omega t)
\]
\[
= -\frac{V_s^2}{2\pi \omega L} \int \sin(\omega t) \cos(\omega t) d(\omega t)
\]
\[
= 0 \text{ (of course)}
\] (3.5)

Mathematically, it is because \( V \) and \( i \) are orthogonal. While we draw no real power, we still draw current.

\[
i_{RMS} = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} i^2(\omega t) d(\omega t)}
\]
\[
= \frac{V_s}{\sqrt{2\omega L}}
\] (3.6)

@115V, 60Hz, \( L \leq 20\text{mH} \) \( \rightarrow i_{RMS} \geq 15\text{A} \) (3.7)

So we still will blow the fuse (to protect the wall wiring), even though we do not
draw any real power at the output! (some power dissipated in $R_{wire}$). In this case we are not utilizing the source well.

**Power Factor**

To provide a measure of the utilization of the source we define Power Factor.

\[
P.F. = \frac{< P >}{V_{RMS}i_{RMS}} = \frac{\text{Real Power}}{\text{Apparent Power}}
\]  

(3.8)

For a resistor $< P >= V_{RMS}i_{RMS} \rightarrow P.F. = 1$ best utilization. For a inductor $< P >= 0 \rightarrow P.F. = 0$ worst utilization.

Consider a rectifier drawing some current waveform,

\[\text{Figure 3.3: Rectifier}\]

Express $i(t)$ as a Fourier series:

\[i(t) = \sum_{n=0}^{\infty} i_n \sin(n\omega t + \phi_n) \text{ Sum of weighted shifted sinusoids} \quad (3.9)\]

Note: $i_{RMS} = \sqrt{\frac{1}{2} i_1^2 + \frac{1}{2} i_2^2 + \cdots + \frac{1}{2} i_n^2 + \cdots}$

$< P > = \frac{1}{2\pi} \int_{2\pi} V(t)i(t)d(\omega t)$
\[ <P> = \frac{1}{2\pi} \int_{2\pi} V_s \sin(\omega t) \sum_n i_n \sin(n\omega t + \phi_n) \]
\[ = \sum_{n=0}^{\infty} \frac{1}{2} \int_{2\pi} V_s i_n \sin(\omega t) \sin(n\omega t + \phi_n) \] (3.10)

By orthogonality all terms except fundamental drop out.

\[ <P> = \frac{1}{2} \int_{2\pi} V_s i_1 \sin(\omega t) \sin(\omega t + \phi_1) \]
\[ = \frac{V_s i_1}{2} \cos \phi_1 \]
\[ = V_{s, RMS} i_{1, RMS} \cos \phi_1 \] (3.11)

So the only current that contributes to real power is the fundamental component in phase with the voltage.

\[ P.F. = \frac{V_{\text{RMS}} i_{1, \text{RMS}}}{V_{\text{RMS}} i_{\text{RMS}}} \cos \phi_1 \]
\[ = \frac{i_{1, \text{RMS}}}{i_{\text{RMS}}} \cos \phi_1 \] (3.12)

We can break down into two factors:

\[ P.F. = \left( \frac{i_{1, \text{RMS}}}{i_{\text{RMS}}} \right) \cdot \cos \phi_1 \]
\[ = k_d(\text{distortion factor}) \cdot k_\theta(\text{displacement factor}) \] (3.13)

- \( k_d \), distortion factor \((\leq 1)\) tells us how much the utilization of the source is reduced because of harmonic currents that do not contribute to power.
• $k_d$, displacement factor ($\leq 1$) tells us how much utilization is reduced due to phase shift between the voltage and fundamental current.

Total Harmonic Distortion (THD)

Consider another measure of distortion: Total Harmonic Distortion (THD).

$$THD = \sqrt{\sum_{n\neq 1} i_n^2 / i_1^2}$$

(3.14)

This measure the RMS of the harmonics normalized to the RMS of the fundamental (square root of the power ratio). Distortion factor and THD are related:

$$THD^2 = \frac{i_{RMS}^2}{i_{1,RMS}^2} - 1$$

$$\frac{i_{RMS}^2}{i_{1,RMS}^2} = 1 + THD^2$$

$$\frac{i_{RMS}}{i_{1,RMS}} = \sqrt{1 + THD^2}$$

$$k_d = \sqrt{\frac{1}{1 + THD^2}}$$

(3.15)

Example:

$$V = V_s \sin(\omega t)$$
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\[ i(t) = \text{square wave} \begin{cases} 
  i_n &= \frac{4}{\pi n} \left( \frac{i_{pk}}{2} \right) \\
  i_0 &= i_{ave} = \frac{1}{2} i_{pk} 
\end{cases} \]

\[ \text{THD} = 121\% \]

\[ k_d = \frac{i_{pk}}{2} \cdot \frac{4}{\pi} \cdot \frac{1}{\sqrt{2}} \]

\[ = \frac{2}{\pi} \]

\[ P.F. = 0.63 \quad (3.16) \]

Figure 3.4: Example

(Passive) Power Factor Compensation (KSV: Section 3.4.1)

Let’s focus on the displacement factor component of power factor. For simplicity, let’s assume a linear load (e.g. R-L) so that voltages and currents are sinusoidal.

For sinusoidal \( V \) and \( i \):

\[ P.F. = \frac{\langle P \rangle}{V_{RMS} i_{RMS}} = \cos \phi \quad (3.17) \]

\( \phi \) is the power factor angle:

- Leading \( \phi < 0 \) Capacitive
- Lagging \( \phi > 0 \) Inductive
Real power:

\[ P = V_{RMS} I_{RMS} \cos \phi \]  \hspace{1cm} (3.18)

Define reactive power as:

\[ Q = V_{RMS} I_{RMS} \sin \phi \]  \hspace{1cm} (3.19)

Figure 3.5: Reactive Power

In vector form \( \vec{S} = P + jQ \). In phaser form \( \vec{V}, \vec{i} \rightarrow \vec{S} = < VI^* > \)

\[ \text{units} \]

Apparent Power \( S = \| \vec{S} \| = V_{RMS} I_{RMS} \quad \text{VA} \)

Average Power \( Re\{S\} = P = V_{RMS} I_{RMS} \cos \phi \quad \text{W} \)

Reactive Power \( Im\{S\} = Q = V_{RMS} I_{RMS} \sin \phi \quad \text{VAR} \)

We can use these results to help adjust the displacement factor of a system. (make \( Q_{net} \rightarrow 0 \)).
Suppose we have an R-L load (e.g. an induction machine):

\[ i(t) = \frac{V_s}{\sqrt{\omega^2L^2 + R^2}} \cos(\omega t - \arctan(\frac{\omega L}{R})) \]

since \( S \triangleq VI^* \)

voltage-current phase \( \phi = \arctan(\frac{\omega L}{R}) \)

\( P.F. = \cos(\arctan(\frac{\omega L}{R})) \)

\[ = \frac{R}{\sqrt{R^2 + \omega^2L^2}} < 1 \quad (3.20) \]

We can add some additional reactive load to balance out and give net unity power factor.

\[ S = V_{RMS}I_{RMS} \]

\[ = \frac{V_s^2}{2\sqrt{\omega^2L^2 + R^2}} \quad (3.21) \]

\[ P = S \cos \phi \]

\[ = V_{RMS}I_{RMS} \cos \phi \]
\begin{equation}
\frac{V_s^2 R}{2(\omega^2 L^2 + R^2)} \quad (3.22)
\end{equation}

\begin{equation}
\begin{align*}
qQ &= jS \sin \phi \\
&= jV_{RMS}I_{RMS} \sin \phi \\
&= j \frac{\omega L V_s^2}{2(\omega^2 L^2 + R^2)} \quad (3.23)
\end{align*}
\end{equation}

So we have real and reactive power.

Suppose we add a capacitor in parallel:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{ capacitor.png}
\caption{Capacitor}
\end{figure}

\begin{align*}
Z_c &= \frac{1}{j\omega C} \\
&= \frac{1}{\omega C} e^{-j\frac{\pi}{2}} \\
\frac{1}{Z_c} &= \omega C e^{j\frac{\pi}{2}} \quad (3.24)
\end{align*}

\begin{align*}
V_{\text{phase}} - i_{\text{phase}} &= -90^\circ \\
i' &= V_s \omega C \sin(\omega t + \frac{\pi}{2}) \quad (3.25) \\
S' &= V_{RMS}I_{RMS} \\
&= \frac{1}{2}V_s^2 \omega C \quad (3.26) \\
P' &= 0 \quad (3.27)
\end{align*}
\[ Q' = -j\frac{1}{2}V_s^2\omega C \] 

(3.28)

So by placing the capacitor in parallel:

\[ S = P + jQ + jQ' \]

make \( jQ \) and \( jQ' \) cancel: \( Q + Q' = 0 \)

\[ \frac{j\omega L V_s^2}{2(\omega^2 L^2 + R^2)} - j\frac{1}{2}V_s^2\omega C = 0 \]

\[ C = \frac{L}{\omega^2 L^2 + R^2} \] 

(3.29)

Example:

\[ \omega = 377 \text{RAD/sec (\omega HZ)} \]

\[ R = 1 \Omega \]

\[ L = 2.7 \text{mH} \]

\[ \Rightarrow C = 1.32 \text{mF} \]
If we know our load, we can add reactive elements to compensate so that no displace-ment factor reduction of line utilization occurs. Real, reactive power definitions are useful to help us do this. This does not help with distortion factor.