Introduction to Simulation - Lecture 8

1-D Nonlinear Solution Methods

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Outline

• Nonlinear Problems
  – Struts and Circuit Example

• Richardson and Linear Convergence
  – Simple Linear Example

• Newton’s Method
  – Derivation of Newton
  – Quadratic Convergence
  – Examples
  – Global Convergence
  – Convergence Checks
Nonlinear problems

Strut Example

Given: \( x_0, y_0, x_1, y_1, W \)
Find: \( x_2, y_2 \)

Need to Solve

\[
\sum f_x = 0 \quad \sum f_y + W = 0
\]
Nonlinear Problems

Struts Example

Reminder: Strut Forces

\[ f = EA_c \frac{L_0 - L}{L_0} = \varepsilon (L_0 - L) \]

\[ f_x = \frac{x_1}{L} f \]

\[ f_y = \frac{y_1}{L} f \]

\[ L = \sqrt{x_1^2 + y_1^2} \]
Nonlinear problems

Strut Example

\[ L_1 = \sqrt{(x_2 - x_0)^2 + (y_2 - y_0)^2} \]
\[ L_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
\[ f_{1x} = \frac{x_2 - x_0}{L_1} \varepsilon (L_0 - L_1) \]
\[ f_{2x} = \frac{x_2 - x_1}{L_2} \varepsilon (L_0 - L_2) \]
\[ \sum f_{1x} + f_{2x} = 0 \]
\[ \sum f_{1y} + f_{2y} + W = 0 \]
Nonlinear problems

Strut Example

Why Nonlinear?

The strut forces change in both magnitude and direction

\[
\frac{y_2 - y_1}{L_2} \varepsilon (L_o - L_2) + \frac{y_2 - y_0}{L_1} \varepsilon (L_o - L_1) + W = 0
\]

Pull Hard on the Struts

The strut forces change in both magnitude and direction
Nonlinear problems

Circuit Example

\[ I_r - \frac{1}{10} V_r = 0 \]

\[ I_d - I_s \left( e^{\frac{V_d}{V_i}} - 1 \right) = 0 \]

Need to Solve

\[ I_d + I_r = 0 \]

\[ I_{v_{src}} - I_r = 0 \]
Solve Iteratively

Nonlinear problems

Hard to find analytical solution for \( f(x) = 0 \)

Solve iteratively

- guess at a solution \( x^0 = x_0 \)
- repeat for \( k = 0, 1, 2, \ldots \)

\[
x^{k+1} = W(x^k)
\]

until \( f(x^{k+1}) \approx 0 \)

Ask

- Does the iteration converge to correct solution?
- How fast does the iteration converge?
Richardson Iteration Definition

Let \( x^{k+1} = x^k + f(x^k) \).

An iteration stationary point is a solution

\[
    x^{k+1} = x^k
\]

\[\Rightarrow f(x^k) = 0\]

\[\Rightarrow x^k = x^* \quad (Solution)\]
Richardson Iteration

Example 1

\[ f(x) = -0.7x + 10 \]

Start with \( x^0 = 0 \)

\[
\begin{align*}
  x^1 &= x^0 + f(x^0) = 10 \\
  x^2 &= x^1 + f(x^1) = 13 \\
  x^3 &= x^2 + f(x^2) = 13.9 \\
  x^4 &= x^3 + f(x^3) = 14.17 \\
  x^5 &= 14.25 \\
  x^6 &= 14.27 \\
  x^7 &= 14.28 \\
  x^8 &= 14.28 \\
\end{align*}
\]

Converged
Richardson Iteration

Example 1

\[ f(x) = -0.7x + 10 \]

\[ \left| x^k - x^* \right| \]
Richardson Iteration

Example 2

\[ f(x) = 2x + 10 \]

Start with \( x_0 = 0 \)

\[
\begin{align*}
x_1 &= x_0 + f(x_0) = 10 \\
x_2 &= x_1 + f(x_1) = 40 \\
x_3 &= x_2 + f(x_2) = 130 \\
x_4 &= x_3 + f(x_3) = 400
\end{align*}
\]

No convergence!
Richardson Iteration

Iteration Equation

\[ x^{k+1} = x^k + f(x^k) \]

Exact Solution

\[ x^* = x^* + f(x^*) \]

Computing Differences

\[ x^{k+1} - x^* = x^k - x^* + f(x^k) - f(x^*) \]

Need to Estimate
Convergence

Richardson Iteration

Mean Value Theorem

\[ f(v) - f(y) = \frac{\partial f(\tilde{v})}{\partial x}(v - y) \quad \tilde{v} \in [v, y] \]
**Richardson Iteration**

**Convergence**
Use MVT

**Iteration Equation**
\[ x^{k+1} = x^k + f(x^k) \]

**Exact Solution**
\[ x^* = x^* + f(x^*) = 0 \]

**Computing Differences**
\[ x^{k+1} - x^* = x^k - x^* + f(x^k) - f(x^*) \]
\[ = \left(1 + \frac{\partial f(x)}{\partial x}\right)(x^k - x^*) \]
If
\[ \left| 1 + \frac{\partial f(\tilde{x})}{\partial x} \right| \leq \gamma < 1 \] for all \( \tilde{x} \) s.t. \( |\tilde{x} - x^*| < \delta \)

And
\[ |x^0 - x^*| < \delta \]

Then
\[ |x^{k+1} - x^*| \leq \gamma |x^k - x^*| \]

Or
\[ \lim_{k \to \infty} |x^{k+1} - x^*| = \lim_{k \to \infty} \gamma^k |x^0 - x^*| = 0 \]

Linear Convergence
Richardson Iteration

Example 1

\[ f(x) = -0.7x + 10 \]

\[ |x^k - x^*| \]
Richardson Iteration

Problems

• Convergence is only linear
• \( x, f(x) \) not in the same units:
  – \( x \) is a voltage, \( f(x) \) a current in circuits
  – \( x \) is a displacement, \( f(x) \) a force in struts
  – Adding 2 different physical quantities
• But a Simple Algorithm
  – Just calculate \( f(x) \) and update
Another approach

Newton’s method

From the Taylor series about solution

\[
0 = f(x^*) \approx f(x^k) + \frac{df}{dx}(x^k)(x^* - x^k)
\]

Define iteration

Do \( k = 0 \) to \( \ldots \)

\[
x^{k+1} = x^k - \left[ \frac{df}{dx}(x^k) \right]^{-1} f(x^k)
\]

if \( \left[ \frac{df}{dx}(x^k) \right]^{-1} \) exists

until convergence
Newton’s Method

Graphically

\[ f(x^0) + f'(x^0)(x - x^0) \]

\[ f(x) \]
Newton’s Method

EXAMPLE: $f(x) = x^3 - 2$, $x^* = \sqrt[3]{2} \approx 1.259921$

| $k$ | $x^k$     | $|x^k - x^*|$ |
|-----|-----------|-------------|
| 0   | 10.0      | 8.740       |
| 1   | 6.6733333 | 5.413       |
| .   | .         | .           |
| 8   | 1.261665  | 1.744e-03   |
| 9   | 1.259924  | 2.410e-06   |
| 10  | 1.259921  | 4.609e-12   |

Asymptotically,

$|x^{k+1} - x^*| \approx C|x^k - x^*|^\alpha$

$C = 0.7951$
$\alpha = 2.000$ Quadratic
Newton’s Method

Example

\[ |x^k - x^*| \]

\[ f(x) = x^3 - 2 \]

Newton’s method

Iteration k
Newton’s Method

$$0 = f(x^*) = f(x^k) + \frac{df}{dx}(x^k)(x^* - x^k) + \frac{d^2f}{dx^2}(\tilde{x})(x^* - x^k)^2$$

some $\tilde{x} \in [x^k, x^*]$

Mean Value theorem truncates Taylor series

But

$$0 = f(x^k) + \frac{df}{dx}(x^k)(x^{k+1} - x^k)$$

by Newton definition
Newton’s Method

Subtracting
\[ \frac{df}{dx}(x^k)(x^{k+1} - x^*) = \frac{d^2 f}{d^2 x}(\tilde{x})(x^k - x^*)^2 \]

Dividing through
\[ (x^{k+1} - x^*) = \left[ \frac{df}{dx}(x^k) \right]^{-1} \frac{d^2 f}{d^2 x}(\tilde{x})(x^k - x^*)^2 \]

Suppose
\[ \left| \left[ \frac{df}{dx}(x) \right]^{-1} \frac{d^2 f}{d^2 x}(x) \right| \leq L \quad \text{for all} \quad x \]

then
\[ |x^{k+1} - x^*| \leq L |x^k - x^*|^2 \]

Convergence is quadratic if L is bounded
Newton’s Method

Convergence

Example 1

\[ f(x) = x^2 - 1 = 0, \quad \text{find} \quad x \quad (x^* = 1) \]

\[ \frac{df}{dx}(x^k) = 2x^k \]

\[ 2x^k(x^{k+1} - x^k) = -\left( (x^k)^2 - 1 \right) \]

\[ 2x^k(x^{k+1} - x^*) + 2x^k(x^* - x^k) = -\left( (x^k)^2 - (x^*)^2 \right) \]

or \[ (x^{k+1} - x^*) = \frac{1}{2x^k} (x^k - x^*)^2 \]

Convergence is quadratic
Newton’s Method

Example 2

\[ f(x) = x^2 = 0, \quad x^* = 0 \]

\[ \frac{df}{dx}(x^k) = 2x^k \]

\[ \Rightarrow 2x^k (x^{k+1} - 0) = (x^k - 0)^2 \]

\[ x^{k+1} - 0 = \frac{1}{2} (x^k - 0) \quad \text{for } x^k \neq x^* = 0 \]

\[ \text{or } (x_{k+1} - x^*) = \frac{1}{2} (x_k - x^*) \]

Convergence is linear

Note: \( \left( \frac{df}{dx} \right)^{-1} \) not bounded away from zero
Newton’s Method

Convergence

Examples 1, 2

\[ f(x) = x^2 - 1 \]

\[ f(x) = x \]

| Residual_k |

\[ \text{Iteration } k \]
Newton’s Method

Convergence

Suppose
\[
\left| \frac{d}{dx} \left( \frac{df}{dx}(x) \right)^{-1} \frac{d^2 f}{d^2 x}(x) \right| \leq L \quad \text{for all } x
\]

if \( L \left| x_0 - x^* \right| \leq \gamma < 1 \)

then \( x_k \) converges to \( x^* \)

Proof
\[
\left| x_1 - x^* \right| \leq L \left| (x_0 - x^*) \right| \left| x_0 - x^* \right|
\]
\[
\Rightarrow \left| x_1 - x^* \right| \leq \gamma \left| x_0 - x^* \right|
\]
\[
\Rightarrow \left| x_2 - x^* \right| \leq L \gamma \left| x_0 - x^* \right| \left| x_1 - x^* \right|
\]
\[
\Rightarrow \left| x_2 - x^* \right| \leq L \gamma \left| x_0 - x^* \right| \left| x_1 - x^* \right|
\]
\[
\text{or} \left| x_2 - x^* \right| \leq \gamma^2 \left| x_1 - x^* \right| \leq \gamma^3 \left| x_0 - x^* \right|
\]
\[
\Rightarrow \left| x_3 - x^* \right| \leq \gamma^4 \left| x_2 - x^* \right| \leq \gamma^7 \left| x_0 - x^* \right|
\]
**Theorem**

If \( L \) is bounded (\( \frac{df}{dx} \) bounded away from zero; \( \frac{d^2f}{dx^2} \) bounded) then Newton's method is guaranteed to converge given a "close enough" guess.

**Always converges?**
Newton’s Method

Convergence Depends on a Good Initial Guess
Newton’s Method

Convergence Checks

Need a "delta-x" check to avoid false convergence

\[ \| x^{k+1} - x^k \| > \varepsilon_{x_a} + \varepsilon_{x_r} \| x^{k+1} \| \]

\[ \| f(x^{k+1}) \| < \varepsilon_{f_a} \]
Newton’s Method

Convergence Checks

Also need an \( f(x) \) check to avoid false convergence

\[
\left\| f\left(x^{k+1}\right) \right\| > \varepsilon_{f_a}
\]

\[
\left\| x^{k+1} - x^k \right\| < \varepsilon_{x_a} + \varepsilon_{x_r} \left\| x^{k+1} \right\|
\]
Summary

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  – Quadratic Convergence
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