Problem 1.1 (OSB 2.1)

Answers are in the back of the book but have a few typos:

(d) The system is causal when $n_0 \geq 0$, not when $n_0 \leq 0$.

(h) (not assigned but for your benefit) The system is also causal.

Problem 1.2 (OSB 2.6)

(a)

$$y[n] - \frac{1}{2} y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

$$Y(e^{j\omega}) \left[ 1 - \frac{1}{2} e^{-j\omega} \right] = X(e^{j\omega}) \left[ 1 + 2e^{-j\omega} + e^{-j2\omega} \right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2} e^{-j\omega}}$$

(b)

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2} e^{-j\omega} + e^{-j3\omega}}{1 + \frac{3}{4} e^{-j\omega} + \frac{3}{2} e^{-j2\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$Y(e^{j\omega}) \left[ 1 + \frac{1}{2} e^{-j\omega} + \frac{3}{4} e^{-j2\omega} \right] = X(e^{j\omega}) \left[ 1 - \frac{1}{2} e^{-j\omega} + e^{-j3\omega} \right]$$

$$y[n] + \frac{1}{2} y[n-1] + \frac{3}{4} y[n-2] = x[n] - \frac{1}{2} x[n-1] + x[n-3]$$

Problem 1.3 (OSB 2.11)

We can write $x[n]$ as a sum of exponentials and compute the response of the system to each exponential:
For $x[n] = \sin \left( \frac{\pi n}{4} \right)$,

$$x[n] = \sin \left( \frac{\pi n}{4} \right) = \frac{1}{2j} \left[ e^{j\frac{\pi n}{4}} - e^{-j\frac{\pi n}{4}} \right]$$

Response to $e^{j\frac{\pi n}{4}}$:

$$H(e^{j\frac{\pi n}{4}})e^{j\frac{\pi n}{4}} = \left[ \frac{1 - e^{-j\frac{2\pi}{4}}}{1 + \frac{1}{2}e^{-j\frac{\pi}{4}}} \right] e^{j\frac{\pi n}{4}} = 2(1+j)e^{j\frac{\pi n}{4}} = 2\sqrt{2}e^{j\frac{\pi n}{4}}e^{j\frac{\pi n}{4}}$$

Response to $e^{-j\frac{\pi n}{4}}$:

$$H(e^{-j\frac{\pi n}{4}})e^{-j\frac{\pi n}{4}} = \left[ \frac{1 - e^{j\frac{2\pi}{4}}}{1 + \frac{1}{2}e^{j\frac{\pi}{4}}} \right] e^{-j\frac{\pi n}{4}} = 2(1-j)e^{-j\frac{\pi n}{4}} = 2\sqrt{2}e^{-j\frac{\pi n}{4}}e^{-j\frac{\pi n}{4}}$$

$$y[n] = \frac{1}{2j} \left[ H(e^{j\frac{\pi n}{4}})e^{j\frac{\pi n}{4}} - H(e^{-j\frac{\pi n}{4}})e^{-j\frac{\pi n}{4}} \right] = \frac{1}{2j} \left[ 2\sqrt{2} \left( e^{j\frac{\pi n}{4}} - e^{-j\frac{\pi n}{4}} \right) \right]$$

$$y[n] = 2\sqrt{2} \sin \left( \frac{\pi(n+1)}{4} \right)$$

Note: Answer in the back of the book has a typo.

For $x[n] = \sin \left( \frac{7\pi n}{4} \right)$, we have to map the frequency $\frac{7\pi}{4}$ into the $-\pi$ to $\pi$ range where $H(e^{j\omega})$ is defined.

$$x[n] = \sin \left( \frac{7\pi n}{4} \right) = \sin \left( -\frac{\pi n}{4} \right) = -\sin \left( \frac{\pi n}{4} \right)$$

Then from above we have:

$$y[n] = -2\sqrt{2} \sin \left( \frac{\pi(n+1)}{4} \right)$$

**Problem 1.4 (OSB 2.55)**

Yes. Suppose $x_1[n] = \cos(\omega n)$ and $x_2[n] = \cos((\omega + 2\pi)n)$. Then $x_1[n] = x_2[n]$ and the inputs are identical. All three systems behave deterministically, so the intermediate signals and the respective outputs $A_1$ and $A_2$ will be identical. Thus $A$ will be periodic in $\omega$ (with period $2\pi$). Recall that all distinct frequencies in discrete time fall within a continuous range of $2\pi$. More generally for a sinusoidal input, the output of the overall system will be periodic in $\omega$ regardless of the systems in between the input and output (since the inputs $x_1[n]$ and $x_2[n]$ above will always be equal).
Problem 1.5 (OSB 3.4)

(a) If the Fourier transform is known to exist, then the ROC must include the unit circle. Thus, the ROC of $X(z)$ is $\frac{1}{3} < |z| < 2$ and therefore $x[n]$ is a two-sided sequence.

(b) Two: $\frac{1}{3} < |z| < 2$ and $2 < |z| < 3$.

(c) No. Stability requires the ROC to include the unit circle and causality requires the ROC to extend outward from the outermost pole and include $z = \infty$. It is not possible to satisfy both conditions at once, so it is not possible to have both stability and causality.

Problem 1.6 (OSB 3.9)

(a) For $H(z)$ to be causal the ROC must extend outward from the outermost pole and include $z = \infty$. The ROC is thus $|z| > \frac{1}{3}$.

(b) Yes, the system is stable since the ROC includes the unit circle.

(c) $y[n] = -\frac{1}{3} \left( -\frac{1}{4} \right)^n u[n] - \frac{4}{3} (2)^n u[-n-1]$ 

$$Y(z) = -\frac{\frac{1}{3}}{1 + \frac{1}{4} z^{-1}} + \frac{\frac{4}{3}}{1 - 2z^{-1}} = \frac{1 + z^{-1}}{(1 + \frac{1}{4} z^{-1})(1 - 2z^{-1})}$$

Since the first term of $y[n]$ is right-sided, the corresponding ROC constraint is $|z| > \frac{1}{3}$; likewise, the second term being left-sided leads to the ROC constraint $|z| < 2$. Therefore, the ROC for $Y(z)$ is $\frac{1}{3} < |z| < 2$.

$$X(z) = \frac{Y(z)}{H(z)} = \frac{(1 + z^{-1})}{(1 + \frac{1}{4} z^{-1})} \frac{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{4} z^{-1})}{(1 + z^{-1})} = \frac{1 - \frac{1}{2} z^{-1}}{1 - 2z^{-1}}$$

Only the pole at $z = 2$ remains, so the ROC for $X(z)$ is $|z| < 2$.

(d) $H(z) = \frac{1 + z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{4} z^{-1})} = \frac{2}{1 - \frac{1}{2} z^{-1}} - \frac{1}{1 + \frac{1}{4} z^{-1}}, \quad |z| > \frac{1}{2}$

$$h[n] = 2 \left( \frac{1}{2} \right)^n u[n] - \left( -\frac{1}{4} \right)^n u[n]$$
\[(e)\]
\[H(z) = \frac{1+z^{-1}}{1 - \frac{1}{2} z^{-1} (1 + \frac{1}{4} z^{-1})} = \frac{1+z^{-1}}{1 - \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}} = \frac{Y(z)}{X(z)}\]

\[Y(z) - \frac{1}{4} z^{-1} Y(z) - \frac{1}{8} z^{-2} Y(z) = X(z) + z^{-1} X(z)\]

\[y[n] - \frac{1}{4} y[n-1] - \frac{1}{8} y[n-2] = x[n] + x[n-1].\]

**Problem 1.7 (OSB 3.40)**

(a) Translating the block diagram into z-transforms,

\[\left[ X(z) - W(z) \right] H(z) + E(z) = W(z)\]

\[W(z) = \frac{H(z)}{1 + H(z)} X(z) + \frac{1}{1 + H(z)} E(z)\]

\[H_1(z) = \frac{H(z)}{1 + H(z)} \quad \text{and} \quad H_2(z) = \frac{1}{1 + H(z)}\]

(b) \(H_1(z) = z^{-1}\)
\(H_2(z) = 1 - z^{-1}\)

(c) \(H(z)\) has a pole at \(z = 1\) on the unit circle so it is not stable. But both \(H_1(z)\) and \(H_2(z)\) have poles at \(z = 0\) and are stable (\(H(z)\) is causal).

**Problem 1.8 (OSB 3.46)**

(a) The ROC must contain the unit circle in order for \(y[n]\) to be stable, thus the ROC for \(Y(z)\) is: \(\frac{1}{2} < |z| < 2\).

(b) \(y[n]\) is two-sided.

(c) Again the ROC must contain the unit circle for \(x[n]\) to be stable, thus the ROC of \(X(z)\) is: \(|z| > \frac{3}{4}\).

(d) Yes, \(x[n]\) is causal, since the ROC extends outward from the outermost pole and includes \(z = \infty\).
(e) Since \( x[n] \) is causal, we can use the initial-value theorem:

\[
x[0] = \lim_{z \to \infty} X(z) = 0.
\]

The limit goes to zero because \( X(z) \) has a zero at \( z = \infty \). Rational z-transforms have an equal number of poles and zeroes if we include the singularities at infinity. We can also verify the zero at \( z = \infty \) by writing the mathematical expression for \( X(z) \):

\[
X(z) = \frac{Az^{-1} (1 - \frac{1}{4} z^{-1})}{(1 + \frac{3}{4} z^{-1}) (1 - \frac{1}{2} z^{-1})},
\]

from which we see that as \( z \to \infty \), \( X(z) \to 0 \).

(f) From the pole-zero diagrams for \( Y(z) \) and \( X(z) \) we have the following:

- Poles of \( Y(z) \): \( z = \frac{1}{2} \) and \( z = 2 \).
- Zeroes of \( Y(z) \): \( z = 0 \) and \( z = \frac{1}{4} \).
- Poles of \( X(z) \): \( z = -\frac{3}{4} \) and \( z = \frac{1}{2} \).
- Zeroes of \( X(z) \): \( z = \infty \) and \( z = \frac{1}{4} \).

\[
H(z) = \frac{Y(z)}{X(z)}
\]

Inverting \( X(z) \) will turn the poles of \( X(z) \) into zeroes and vice versa. As a result we have pole-zero cancellation at \( z = \frac{1}{2} \) and \( z = \frac{1}{4} \). \( H(z) \) has zeroes at \( z = 0 \) and \( z = -\frac{3}{4} \), and poles at \( z = 2 \) and \( z = \infty \). Its ROC is \(|z| < 2\).

Alternatively, we could have determined \( H(z) \) from the mathematical expressions for \( Y(z) \) and \( X(z) \):

\[
Y(z) = \frac{B (1 - \frac{1}{4} z^{-1})}{(1 - \frac{1}{2} z^{-1})(1 - 2 z^{-1})}
\]

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{B (1 - \frac{1}{4} z^{-1})}{(1 - \frac{1}{2} z^{-1})(1 - 2 z^{-1})} \frac{(1 + \frac{3}{4} z^{-1}) (1 - \frac{1}{2} z^{-1})}{A z^{-1} (1 - \frac{1}{4} z^{-1})} = \frac{C (1 + \frac{3}{4} z^{-1})}{z^{-1} (1 - 2 z^{-1})},
\]

where \( C = \frac{B}{A} \).

(g) Yes. \( h[n] \) is anti-causal, since the ROC of \( H(z) \) extends inward from the innermost pole and does include the origin \((z = 0)\).
Problem 1.9

(a) (i) \( 1/T_1 = 1/T_2 = 2 \times 10^4 \)

\[ X(e^{j\omega}) \]

\[ Y(e^{j\omega}) \]

\[ Y_c(j\Omega) \]

(ii) \( 1/T_1 = 4 \times 10^4, 1/T_2 = 10^4 \)

\[ X(e^{j\omega}) \]
(iii) $1/T_1 = 10^4$, $1/T_2 = 3 \times 10^4$
(b) From the figure below, the only portion of the spectrum which remains unaffected by the aliasing is $|\omega| < \pi/3$. So if we choose $\omega_c < \pi/3$, the overall system is LTI with a frequency response of:

$$H_c(j\Omega) = \begin{cases} 1 & \text{for } |\Omega| < \omega_c \times 6 \times 10^3 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 1.10

$y[n] = y_2[n]$

Justification:

The input signal $x[n]$ is made up of three narrow-band pulses: pulse-1 is a low-frequency pulse (whose peak is around $0.12\pi$ radians), pulse-2 is a higher-frequency pulse ($0.3\pi$ radians), and pulse-3 is the highest-frequency pulse ($0.5\pi$ radians).

Let $H(e^{j\omega})$ be the frequency response of Filter A. We read off the following values from the frequency response magnitude and group delay plots:

$$|H(e^{j(0.12\pi)})| \approx 1.8$$
$$|H(e^{j(0.3\pi)})| \approx 1.7$$
$$|H(e^{j(0.5\pi)})| \approx 0$$
$$\tau_g(0.12\pi) \approx 40 \text{ samples}$$
$$\tau_g(0.3\pi) \approx 80 \text{ samples}$$

From these values, we would expect pulse-3 to be totally absent from the output signal $y[n]$. Pulse-1 will be scaled up by a factor of 1.8 and its envelope delayed by about 40 samples. Pulse-2 will be scaled up by a factor of 1.7 and its envelope delayed by about 80 samples. The correct output is thus $y_2[n]$. 
Problem 1.11

Uniform Distribution: \( f_x(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases} \)

\( \mu = E[X] = \int_{-\infty}^{\infty} x f_x(x) dx = \int_{0}^{\Delta} \frac{1}{\Delta} x dx = \frac{\Delta}{2} \)

\( \sigma^2 = E[(X - E[X])^2] = E[X^2] - \mu^2 = \int_{0}^{\Delta} \frac{1}{\Delta} x^2 dx - \left(\frac{\Delta}{2}\right)^2 = \frac{\Delta^2}{3} - \frac{\Delta^2}{4} = \frac{\Delta^2}{12} \)

Problem 1.12

(a) \( R_{yx}[m] = R_{xx}[m] * h[m] \), but \( R_{xy}[m] = R_{yx}[-m] \) and \( R_{xx}[m] = R_{xx}[-m] \). Thus \( R_{xy}[m] = R_{xx}[m] * h[-m] \). Since \( R_{xx}[m] = \delta[m] \),

\( R_{xy}[m] = h[-m] = \begin{cases} 1 & m = -2, -1, 0 \\ 0 & \text{otherwise} \end{cases} \)

Then \( R_{yy}[m] = R_{xx}[m] * h[m] * h[-m] = \begin{cases} 1 & m = -2, 2 \\ 2 & m = -1, 1 \\ 3 & m = 0 \\ 0 & \text{otherwise} \end{cases} \)

(b)

\( P_{xx} = 1 \)

\( P_{yy} = 3 + 2e^{j\omega} + 2e^{-j\omega} + e^{2j\omega} + e^{-2j\omega} \)

\( = 3 + 4 \cos \omega + 2 \cos (2\omega) \).

Problem 1.13 (OSB 4.5)

Answers are in the back of the book.

Problem 1.14 (OSB 2.89)

(a)

\( E\{x[n]x[n]\} = \phi_{xx}[0] \)

\( = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega}) d\omega \)
(b) \[
\Phi_{xx}(e^{j\omega}) = \Phi_{ww}(e^{j\omega})|H(e^{j\omega})|^2 \\
= \sigma_w^2 \frac{1}{1 - \cos(\omega) + 1/4} \\
= \frac{\sigma_w^2}{5/4 - \cos \omega}.
\]

(c) \[
\phi_{xx}[n] = \phi_{ww}[n] * h[n] * h[-n] \\
= \sigma_w^2 \left( \frac{1}{2} \right)^n u[n] * \left( \frac{1}{2} \right)^{-n} u[-n] \\
= \frac{4}{3} \sigma_w^2 \left( \frac{1}{2} \right)^{|n|}.
\]