Problem 7.1

(a) The normal (Yule-Walker) equations are:

\[ \phi_s[i] = \sum_{k=1}^{2} a_k \phi_s[i-k], \quad i = 1, 2, \]

or in matrix form:

\[
\begin{bmatrix}
\phi_s[0] & \phi_s[1] \\
\phi_s[1] & \phi_s[0]
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}
= \begin{bmatrix}
\phi_s[1] \\
\phi_s[2]
\end{bmatrix}.
\]

(b) Let \( s_1[n] = (\frac{1}{3})^n u[n] \) and \( s_2[n] = (\frac{-1}{2})^n u[n] \). We calculate the following auto- and cross-correlations for \( m > 0 \),

\[
\phi_{s1}[m] = \sum_{n=-\infty}^{\infty} s_1[n+m]s_1[n] = \frac{9}{8} \left( \frac{1}{3} \right)^m
\]

\[
\phi_{s2}[m] = \sum_{n=-\infty}^{\infty} s_2[n+m]s_2[n] = \frac{4}{3} \left( \frac{-1}{2} \right)^m
\]

\[
\phi_{s1s2}[m] = \sum_{n=-\infty}^{\infty} s_1[n+m]s_2[n] = \frac{6}{7} \left( \frac{1}{3} \right)^m
\]

\[
\phi_{s2s1}[m] = \sum_{n=-\infty}^{\infty} s_2[n+m]s_1[n] = \frac{6}{7} \left( \frac{-1}{2} \right)^m.
\]

Since \( \phi_s[m] = \phi_{s1}[m] + \phi_{s2}[m] + \phi_{s1s2}[m] + \phi_{s2s1}[m] \)

and \( \phi_s[m] \) is an even function of \( m \), we sum the four correlations and replace \( m \) by \( |m| \):

\[
\phi_s[m] = \frac{111}{56} \left( \frac{1}{3} \right)^{|m|} + \frac{46}{21} \left( \frac{-1}{2} \right)^{|m|}.
\]

Note that the cross-correlations \( \phi_{s1s2}[m] \) and \( \phi_{s2s1}[m] \) by themselves are not even. So \( \phi_s[0] = 4.17, \phi_s[1] = -0.4345 \) and \( \phi_s[2] = 0.7678 \).
(c) Substituting the values of $\phi_s[i]$ into the normal equations and solving for the $a_i$’s results in $a_1 = -0.0859, a_2 = 0.1751$.

(d) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{3} a_k \phi_s[i-k], \; i = 1, 2, 3,$$

or in matrix form:

$$
\begin{bmatrix}
\phi_s[0] & \phi_s[1] & \phi_s[2] \\
\phi_s[1] & \phi_s[0] & \phi_s[1] \\
\phi_s[2] & \phi_s[1] & \phi_s[0]
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= 
\begin{bmatrix}
\phi_s[1] \\
\phi_s[2] \\
\phi_s[3]
\end{bmatrix}.
$$

(e) $\phi_s[3] = -0.2004$.

(f) Substituting the values of $\phi_s[i]$ into the normal equations and solving for the $a_i$’s results in $a_1 = -0.0833, a_2 = 0.1738, a_3 = -0.0146$.

(g) Yes. The signal $s[n]$ is NOT the impulse response of an all-pole filter. Increasing the order will in general update all previous coefficients in an attempt to model $s[n]$ more accurately.

(h) In problem 6.7 $s[n]$ was the impulse response of a two-pole system, which we could model perfectly using a two-pole model. Increasing the order beyond $p = 2$ achieves nothing. In this problem $s[n]$ does not arise from an all-pole system, so it is not generally possible to perfectly model $s[n]$ using only poles. Nevertheless, increasing the order of the all-pole model will yield a closer and closer approximation.

(i) The difference equation for which the impulse response is $s[n]$ is:

$$s[n] = -\frac{1}{6}s[n-1] + \frac{1}{6}s[n-2] + 2\delta[n] + \frac{1}{6}\delta[n-1].$$

For $n \geq 2$ the impulses are zero:

$$s[n] = -\frac{1}{6}s[n-1] + \frac{1}{6}s[n-2].$$

Thus the linear prediction coefficients are $a_1 = -1/6, a_2 = 1/6$. 
Problem 7.2 (OSB 8.31)

We re-write the desired samples of $X(z)$ in terms of the DFT of a second sequence $x_1[n]$. $x[n]$ is only non-zero for $0 \leq n \leq 9$:

$$X(z) = \sum_{n=0}^{9} x[n]z^{-n}$$

$$X(z) \big|_{z=0.5e^{j(2\pi k/10) + (\pi/10)}} = \sum_{n=0}^{9} x[n] \left( 0.5e^{j(2\pi k/10) + (\pi/10)} \right)^{-n}$$

$$= \sum_{n=0}^{9} x[n] \left( 0.5e^{j\pi/10} \right)^{-n} e^{-j(2\pi/10)kn}$$

$$= \sum_{n=0}^{9} x_1[n]e^{-j(2\pi/10)kn}$$

$$= X_1[k], \quad k = 0, 1, \ldots, 9$$

where we have defined $x_1[n] = (2e^{-j\pi/10})^n x[n]$ and we recognize the second last line as the 10-point DFT of $x_1[n]$.

Thus $x_1[n] = (2e^{-j\pi/10})^n x[n]$.

Problem 7.3 (OSB 8.32)

Answer: (c)

Since $y[n]$ is $x[n]$ expanded by 2, the DTFT $Y(e^{j\omega})$ is equal to $X(e^{2j\omega})$, i.e. $X(e^{j\omega})$ with the frequency axis compressed by a factor of 2. The 16-point DFT $Y[k]$ samples $Y(e^{j\omega})$ at frequencies $\omega = \frac{2\pi k}{16}$, $k = 0, 1, \ldots, 15$, which is equivalent to sampling $X(e^{j\omega})$ at frequencies $\omega = \frac{2\pi k}{8}$, $k = 0, 1, \ldots, 15$. But since $X(e^{3j\omega})$ is periodic with period $2\pi$, the last eight samples are the same as the first eight, which in turn are equal to the 8-point DFT $X[k]$. In other words, $Y[k]$ samples $X(e^{j\omega})$ from 0 to $4\pi$ instead of from 0 to $2\pi$. Therefore $Y[k]$ is equal to $X[k]$ repeated back-to-back.
**Problem 7.4** (OSB 8.37)

- For \( g_1[n] \), choose \( H_7[k] \).
  
  We can think of this as a time reversal followed by a shift by \( -N + 1 \).

\[
G_1[k] = \sum_{i=0}^{N-1} g_1[i] W_N^{ik} \quad k = 0, \ldots, N - 1
\]

\[
= \sum_{i=0}^{N-1} x[N - 1 - i] W_N^{ik}
\]

\[
= \sum_{j=0}^{N-1} x[j] W_N^{k(N-1-j)}
\]

\[
= W_N^{k(N-1)} \sum_{j=0}^{N-1} x[j] W_N^{(-k)j}
\]

\[
= W_N^{-k} X[((-k))_N]
\]

\[
= e^{j2\pi k/N} X(e^{-j2\pi k/N})
\]

- For \( g_2[n] \), choose \( H_8[k] \).
  
  This is modulation in time by \( (-1)^n = e^{j\pi n} \), or a shift in the frequency domain by \( \pi \).

\[
G_2[k] = \sum_{i=0}^{N-1} g_2[i] W_N^{ik} \quad k = 0, \ldots, N - 1
\]

\[
= \sum_{i=0}^{N-1} (-1)^i x[i] W_N^{ik}
\]

\[
= \sum_{i=0}^{N-1} x[i] W_N^{i(k+N/2)}
\]

\[
= X[((k + N/2))_N]
\]

\[
= X(e^{j(2\pi/N)(k+N/2)})
\]
• For \( g_3[n] \), choose \( H_3[k] \).

We can interpret the DFT \( X[k] \) as the Fourier series coefficients of \( \tilde{x}[n] \), the periodic replication of \( x[n] \) with period \( N \). Given this interpretation, the DFT \( G_3[k] \) is also equal to the Fourier series of \( \tilde{x}[n] \), but considered as having a period of \( 2N \). However, since \( \tilde{x}[n] \) has a fundamental period of \( N \), the even-indexed coefficients of the length \( 2N \) Fourier series will correspond to the length \( N \) Fourier series coefficients (i.e. \( X[k] \)), while the odd-indexed coefficients will be zero because they are not necessary.

\[
G_3[k] = \sum_{i=0}^{2N-1} g_3[i]W_{2N}^{ik} \quad k = 0, \ldots, 2N - 1
\]

\[
= \sum_{i=0}^{N-1} x[i]W_{2N}^{ik} + \sum_{i=N}^{2N-1} x[i - N]W_{2N}^{ik}
\]

\[
= \sum_{i=0}^{N-1} x[i](W_{2N}^{ik} + W_{2N}^{(i+N)k})
\]

\[
= \sum_{i=0}^{N-1} x[i]W_{2N}^{ik}(1 + W_{2N}^{Nk})
\]

\[
= \sum_{i=0}^{N-1} x[i]W_{2N}^{ik}(1 + (-1)^k)
\]

\[
= X(e^{j2\pi k/2N})(1 + (-1)^k)
\]

\[
= \begin{cases} 
2X(e^{j2\pi k/2N}), & k \text{ even} \\
0, & k \text{ odd}
\end{cases}
\]

• For \( g_4[n] \), choose \( H_6[k] \).

The DFT of \( g_4[n] \) is equal to the DFS of \( \tilde{x}[n] \), the periodic replication of \( x[n] \) with a period of \( N/2 \). In other words, \( g_4[n] \) is \( x[n] \) aliased in time. The DFS of \( \tilde{x}[n] \) is in turn equal to samples of \( X(e^{j\omega}) \) spaced by \( \frac{2\pi}{N/2} = \frac{4\pi}{N} \).
\[ G_4[k] = \sum_{i=0}^{N/2-1} g_4[i]W_N^{ik} \quad k = 0, \ldots, N/2 - 1 \]
\[ = \sum_{i=0}^{N/2-1} (x[i] + x[i + N/2])W_N^{ik} \]
\[ = \sum_{i=0}^{N/2-1} x[i]W_N^{ik} + \sum_{i=0}^{N/2-1} x[i + N/2]W_N^{ik} \]
\[ = \sum_{i=0}^{N/2-1} x[i]W_N^{ik} + \sum_{i=0}^{N/2-1} x[i + N/2]W_N^{ik} \]
\[ = \sum_{i=0}^{N/2-1} x[i]W_N^{ik} + \sum_{j=N/2}^{N-1} x[j]W_N^{jk} \]
\[ = \sum_{i=0}^{N-1} x[i]W_N^{ik} \]
\[ = \sum_{i=0}^{N-1} x[i](e^{-j(4\pi/N)i}) \]
\[ = X(e^{j4\pi k/N}) \]

- For \( g_5[n] \), choose \( H_2[k] \).

We are increasing the length of the signal by zero padding. Thus, we are taking more closely spaced samples of \( X(e^{j\omega}) \).

\[ G_5[k] = \sum_{i=0}^{2N-1} g_5[i]W_{2N}^{ik} \quad k = 0, \ldots, 2N - 1 \]
\[ = \sum_{i=0}^{N-1} x[i]W_{2N}^{ik} \]
\[ = \sum_{i=0}^{N-1} x[i]W_N^{i(k/2)} \]
\[ = X(e^{j2\pi(k/2)/N}) \]
\[ = X(e^{j2\pi k/(2N)}) \]
• For $g_6[n]$, choose $H_1[k]$.

We are expanding $x[n]$ by 2 to form $g_6[n]$. The DTFT of $g_6[n]$ is equal to $X(e^{2j\omega})$, i.e. $X(e^{j\omega})$ with the frequency axis compressed by 2. The $2N$ values of $G_6[k]$ sample two periods of $X(e^{j\omega})$, so the last $N$ samples are equal to the first $N$. Moreover, the first $N$ samples are the same as those in $X[k]$. Thus $G_6[k]$ contains the same frequency samples at $\omega = \frac{2k\pi}{N}$, but now $k$ ranges from 0 to $2N - 1$.

$$G_6[k] = \sum_{i=0}^{2N-1} g_6[i]W_{2N}^{jk} \quad k = 0, \ldots, 2N - 1$$

$$= \sum_{i=0}^{N-1} g[2i]W_{2N}^{2jk} + \sum_{i=0}^{N-1} g[2i + 1]W_{2N}^{(2i+1)k}$$

$$= \sum_{i=0}^{N-1} x[i]W_N^{jk} + 0$$

$$= X(e^{j\frac{2\pi k}{N}})$$

• For $g_7[n]$, choose $H_5[k]$.

We are decimating $x[n]$ by 2, so $X(e^{j\omega})$ is vertically scaled by $\frac{1}{2}$, horizontally stretched by 2, and replicated once. We then obtain samples of the resulting DTFT at frequencies $\omega = \frac{2\pi k}{N/2}$.

$$G_7[k] = \sum_{i=0}^{N/2-1} g_7[i]W_{N/2}^{jk} \quad k = 0, \ldots, N/2 - 1$$

$$= \sum_{i=0}^{N/2-1} x[2i]W_{N/2}^{jk}$$

$$= \sum_{i=0}^{N/2-1} x[2i]W_N^{(2i)k}$$

$$= \sum_{i=0, \text{even}}^{N-1} x[i]W_N^{jk}$$

$$= \sum_{i=0}^{N-1} \frac{1}{2} (x[i] + (-1)^i x[i]) W_N^{jk}$$

$$= \frac{1}{2} \{ X[k] + X[((k + N/2))_N] \}$$

$$= 0.5 \left\{ X(e^{j\frac{2\pi k}{N}}) + X(e^{j\frac{2\pi (k+N/2)}{N}}) \right\}$$
Problem 7.5 (OSB 8.46)

In general, (i) holds if the periodic replication of \( x_i[n] \) is even symmetric about \( n = 0 \); (ii) holds if \( x_i[n] \) has some point of symmetry; (iii) holds if the periodic replication of \( x_i[n] \) has some point of symmetry. Note the subtle difference between (ii) and (iii).

- For \( x_1[n] \):

\[
X_1[k] = 3(1 + W_5^{4k}) + 1(W_5^k + W_5^{3k}) + 2(W_5^{2k})
\]
\[
= 2W_5^{2k}\{3\cos(2k(2\pi/5)) + 1\cos(k(2\pi/5)) + 1\}
\]
\[
X_1(e^{j\omega}) = 2e^{-j2\omega}\{3\cos(2\omega) + \cos\omega + 1\}
\]

(i) No, \( X_1[k] \) is not real for all \( k \).
(ii) Yes, \( X_1(e^{j\omega}) \) has generalized linear phase.
(iii) Yes.

- For \( x_2[n] \):

\[
X_2(e^{j\omega}) = 3 + 2e^{-j2.5\omega}\{1\cos(1.5\omega) + 2\cos(0.5\omega)\}
\]
\[
X_2[k] = 3 + 2W_5^{2.5k}\{\cos(1.5k(2\pi/5)) + 2\cos(0.5k(2\pi/5))\}
\]
\[
= 3 + 2(-1)^k\{1\cos(1.5k(2\pi/5)) + 2\cos(0.5k(2\pi/5))\}
\]

(i) Yes.
(ii) No.
(iii) Yes.

- For \( x_3[n] \):

\[
X_3(e^{j\omega}) = 1 + 2e^{-j2\omega}\{2\cos(2\omega) + 1\cos(1\omega) + 1\}
\]
\[
X_3[k] = 1 + 2W_5^{2k}\{2\cos(2k(2\pi/5)) + 1\cos(k(2\pi/5)) + 1\}
\]

(i) No.
(ii) No.
(iii) No.
**Problem 7.6 (OSB 8.59)**

We want to compute \( R_s[k] = \mathcal{R}(e^{j2\pi k/128}) \), the DTFT of \( r[n] \) sampled at 128 equally spaced frequencies.

Both \( x[n] \) and \( y[n] \) are signals of length 256, so their linear convolution \( r[n] \) has length 511. If we had \( r[n] \), we could calculate \( R_s[k] \) by time-aliasing \( r[n] \) to 128 samples (periodically replicating \( r[n] \) with a period of 128 and extracting one period) and taking the 128-point DFT (module V). However, a linear convolution module is not available, so an alternative way of time-aliasing \( r[n] \) is through circular convolution of \( x[n] \) and \( y[n] \). \( x[n] \) and \( y[n] \) can be circularly convolved by periodically replicating both signals with a period of 128 using module I, performing periodic convolution using module III, and extracting one period of the periodic convolution. The result of this circular convolution is equal to \( r[n] \) time-aliased to 128 samples. However, since the 128-point DFT module (module V) only considers its input between \( n = 0 \) and \( n = 127 \), the explicit extraction of one period is not necessary.

The implementation just described is pictured below. The total cost is 110 units.

![Diagram](image_url)

**Problem 7.7**

(a) Assuming that the overlap-save method is correctly implemented, the output \( y[n] \) of \( S \) can be represented as the linear convolution \( y[n] = x[n] \ast h[n] \). The impulse response \( h[n] \) corresponding to \( H[k] \) is a finite sequence of length 256. However, an ideal frequency-selective filter has an infinite impulse response. Therefore, \( S \) cannot be an ideal frequency-selective filter.

(b) The impulse response \( h[n] \) of \( S \) is the IDFT of \( H[k] \). Since \( H[k] \) is real and even in the circular sense \( (H[k] = H[(−k)_{256}]) \), \( h[n] \) is real.
\[ h[n] = \frac{1}{256} \sum_{k=0}^{255} H[k] W_{256}^{-kn} \quad 0 \leq n \leq 255 \]

\[ = \frac{1}{256} \sum_{k=0}^{31} W_{256}^{-kn} + \frac{1}{256} \sum_{k=225}^{255} W_{256}^{-n(k-256)} \]

\[ = \frac{1}{256} \sum_{k=0}^{31} W_{256}^{-kn} + \frac{1}{256} \sum_{k=-31}^{-1} W_{256}^{-kn} \]

\[ = \frac{1}{256} \sum_{k=-31}^{31} W_{256}^{-kn} \]

\[ = \frac{1}{256} \frac{W_{256}^{31n} - W_{256}^{-32n}}{1 - W_{256}^{-n}} \]

\[ = \frac{1}{256} \frac{W_{256}^{-0.5n} (W_{256}^{31.5n} - W_{256}^{-31.5n})}{W_{256}^{-0.5n} (W_{256}^{0.5n} - W_{256}^{-0.5n})} \]

\[ = \frac{\sin \frac{63\pi n}{256}}{256 \sin \frac{\pi n}{256}} \]

In sum,

\[ h[n] = \begin{cases} 
\sin \frac{63\pi n}{256} & 0 \leq n \leq 255 \\
\frac{256 \sin \frac{\pi n}{256}}{256 \sin \frac{\pi n}{256}} & \text{otherwise}
\end{cases} \]