MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.341 DISCRETE-TIME SIGNAL PROCESSING
Fall 2005

FINAL EXAM

Friday, December 16, 2005 Walker (50-340) 1:30pm–4:30pm

• This is a closed book exam, but three 8.5" × 11" handwritten sheets of notes (both
sides) are allowed.

• Calculators are not allowed.

• Make sure you have all 24 numbered pages of this exam.

• There are 9 problems on the exam.

• The problems are not in order of difficulty. We recommend that you read through all the
problems, then do the problems in whatever order suits you best.

• A correct answer does not guarantee full credit, and a wrong answer does not guarantee
loss of credit. You should clearly but concisely indicate your reasoning and show all
relevant work.

• Please be neat—we can not grade what we can not decipher.

• Only this exam booklet is to be handed in. You may want to work things through on
scratch paper first, and then neatly transfer the work you would like us to look at into
the exam booklet. Let us know if you need additional scratch paper.

• We will again be using the EGRMU grading strategy. This strategy focuses on your level
of understanding of the material associated with each problem. Specifically, when we
grade each part of a problem we will do our best to assess, from your work, your level of
understanding.

• Graded Exams and Final Course Grade:

Graded exams, graded Project IIIs, and final course grades can be picked up from Eric
Strattman (in 36-615 or 36-680, depending on the time of day) on or after WEDNESDAY
morning, December 21. If you would like your graded exam and project mailed to you,
please leave an addressed, stamped envelope with us at the end of the exam. We will use
the envelope as is, so please be sure to address it properly and with enough postage. We
guarantee that we will put it into the proper mailbox, but we can not guarantee anything
beyond that.

OUT OF CONSIDERATION FOR THE 6.341 STAFF, PLEASE
DO NOT ASK FOR GRADES BY PHONE OR EMAIL.
NAME: ________________________________

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Problem 1 (10%) 

[5%] (a) $x[n]$ is a real-valued, causal sequence with discrete-time Fourier transform $X(e^{j\omega})$. Determine a choice for $x[n]$ if the imaginary part of $X(e^{j\omega})$ is given by:

$$\text{Im}\{X(e^{j\omega})\} = 3\sin(2\omega) - 2\sin(3\omega)$$

Work to be looked at and answer:
(b) $y_r[n]$ is a real-valued sequence with discrete-time Fourier transform $Y_r(e^{j\omega})$. The sequences $y_r[n]$ and $y_i[n]$ in Figure 1-1 are interpreted as the real and imaginary parts of a complex sequence $y[n]$, i.e. $y[n] = y_r[n] + jy_i[n]$.

\[
\begin{array}{c}
\text{y} \\
y_r[n] \quad \text{H}(e^{j\omega}) \quad y_i[n] \\
\end{array}
\]

Figure 1-1: System for obtaining $y[n]$ from $y_r[n]$.

Determine a choice for $H(e^{j\omega})$ in Figure 1-1 so that $Y(e^{j\omega})$ is $Y_r(e^{j\omega})$ for negative frequencies and zero for positive frequencies between $-\pi$ and $\pi$, i.e.

\[
Y(e^{j\omega}) = \begin{cases} 
Y_r(e^{j\omega}), & -\pi < \omega < 0 \\
0, & 0 < \omega < \pi
\end{cases}
\]

Work to be looked at and answer:
Problem 2  (12\%)  

Consider the system shown in Figure 2-1, with $H_1(e^{j\omega})$ and $H_2(j\Omega)$ as depicted in Figure 2-2.

![Figure 2-1: System for calculating $y[n]$ from $x[n]$.](image)

![Figure 2-2: Frequency responses of discrete-time LTI filter $H_1$ and continuous-time LTI filter $H_2$.](image)
(a) If $T_1 = T_2 = 10^{-4}$ s and $\Omega_0 = \frac{\pi}{T_1}$, is there a choice of $\omega_0 > 0$ for which the overall system from $x[n]$ to $y[n]$ in Figure 2-1 is a discrete-time LTI system? If so, specify at least one non-zero value of $\omega_0$ for which the system is LTI. Otherwise explain why the system cannot be LTI.

Work to be looked at and answer:

(b) For this part, assume that $T_1 = 10^{-4}$ s and $\omega_0 = \pi$. Determine the most general conditions on $\Omega_0 > 0$ and $T_2$, if any, so that the overall system from $x[n]$ to $y[n]$ in Figure 2-1 is an LTI system.

Work to be looked at and answer:
Problem 3  (12%)

The following are three proposed identities involving compressors and expanders. For each, state whether or not the proposed identity is valid. If your answer is that it is valid, explicitly show why. (In doing this you may make use of the known identities on page 11.) If your answer is no, explicitly give a simple counterexample.

[4%] (a) Proposed identity (a):

```
[Diagram]
```

Work to be looked at and answer:
[4\%] (b) Proposed identity (b):

![Diagram](image)

Work to be looked at and answer:
[4%] (c) Proposed identity (c):

\[ \begin{array}{c}
\uparrow L \\
A \\
\downarrow \\
A \\
\uparrow L
\end{array} \]

where \( L \) is a positive integer, and \( A \) is defined in terms of \( X(e^{j\omega}) \) and \( Y(e^{j\omega}) \) (the respective DTFTs of \( A \)'s input and output) as:

\[
x[n] \xrightarrow{A} y[n]
\]

\[
Y(e^{j\omega}) = (X(e^{j\omega}))^L
\]

Work to be looked at and answer:
Correct identities you may refer to without proof

Noble identities:

Half-sample delay:
Problem 4  (15%)  

We find in a treasure chest a zero-phase FIR filter $h[n]$ with associated DTFT $H(e^{j\omega})$, shown in Figure 4-1.

![Filter response diagram](image)

Figure 4-1: Plot of $H(e^{j\omega})$ from $-\pi \leq \omega \leq \pi$.

The filter is known to have been designed using the Parks-McClellan (PM) algorithm, as summarized on page 15 of this exam. The input parameters to the PM algorithm are known to have been:

- Passband edge $\omega_p$: $0.4\pi$
- Stopband edge $\omega_s$: $0.6\pi$
- Ideal passband gain $G_p$: 1
- Ideal stopband gain $G_s$: 0
- Error weighting function $W(\omega) = 1$

The value of the input parameter $N$ to the algorithm is not known.
Also along with the plot is a fortune in gold, to be claimed by whoever can reproduce the filter with the Parks-McClellan algorithm for these specifications and with an appropriate value of \( N \). Multiple winners share the gold. You are the sole referee and judge.

Two entries have been submitted, each with a different associated value for the input parameter \( N \) to the algorithm.

- **Entry 1**: \( N = N_1 \)
- **Entry 2**: \( N = N_2 > N_1 \)

Both entrants claim to have obtained the required filter using exactly the same Parks-McClellan algorithm and input parameters, except for the value of \( N \).

After inspecting both entries, you determine that they both have DTFTs identical to Figure 4-1, so you deem both of them winners.

[3%] (a) What are possible values for \( N_1 \)?

**Work to be looked at and answer:**

[3%] (b) What are possible values for \( N_2 > N_1 \)?

**Work to be looked at and answer:**
(c) Are the impulse responses $h_1[n]$ and $h_2[n]$ of the two filters submitted by entrants 1 and 2 identical?

Work to be looked at and answer:

(d) Both entrants claim that there can only be one winner, since the alternation theorem requires “uniqueness of the rth-order polynomial.” If your answer to (c) is yes, explain why the alternation theorem is not violated. If your answer is no, show how the two filters, $h_1[n]$ and $h_2[n]$ respectively, relate.

Work to be looked at and answer:
The Parks-McClellan algorithm for zero-phase lowpass filter design:

The algorithm for approximating a lowpass design with a zero-phase PM filter takes as input parameters the following:

- Passband edge frequency $\omega_p$
- Stopband edge frequency $\omega_s$
- Ideal passband gain $G_p$
- Ideal stopband gain $G_s$
- Error weighting function $W(\omega)$
- Length $N$ of the filter response $h[n]$, where

$$h[n] = 0 \quad \text{for} \quad |n| > \frac{N - 1}{2},$$

and $N$ must be odd.

The algorithm returns a filter impulse response which satisfies the alternation theorem, stated below.

**Alternation theorem:** Let $F_P$ denote the closed subset of the disjoint union of closed subsets of the real axis $x$. Then

$$P(x) = \sum_{k=0}^{r} a_k x^k$$

is an $r$th-order polynomial. Also, $D_P(x)$ denotes a given desired function of $x$ that is continuous on $F_P$; $W_P(x)$ is a positive function, continuous on $F_P$, and

$$E_P(x) = W_P(x) [D_P(x) - P(x)]$$

is the weighted error. The maximum error is defined as

$$\|E\| = \max_{x \in F_P} |E_P(x)|.$$

A necessary and sufficient condition that $P(x)$ be the unique $r$th-order polynomial that minimizes $\|E\|$ is that $E_P(x)$ exhibit at least $(r + 2)$ alternations; i.e., there must exist at least $(r + 2)$ values $x_i$ in $F_P$ such that $x_1 < x_2 < \cdots < x_{r+2}$ and such that $E_P(x_i) = -E_P(x_{i+1}) = \pm \|E\|$ for $i = 1, 2, \ldots, (r + 1)$. 
Problem 5  (12%)

[6%] (a) $X(e^{j\omega})$ is the DTFT of the discrete-time signal

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

Find a length-5 sequence $g[n]$ whose 5-point DFT $G[k]$ represents samples of the DTFT of $x[n]$, i.e.

$$g[n] = 0 \text{ for } n < 0, \quad n > 4$$

and

$$G[k] = X(e^{j\frac{2\pi k}{5}}) \text{ for } k = 0, 1, \ldots, 4.$$
(b) Let $w[n]$ be a sequence that is strictly non-zero for $0 \leq n \leq 9$ and zero elsewhere, i.e.

$$w[n] \neq 0, \quad 0 \leq n \leq 9$$
$$w[n] = 0 \quad \text{otherwise}$$

Determine a choice for $w[n]$ such that its DTFT $W(e^{j\omega})$ is equal to $X(e^{j\omega})$ at the frequencies $\omega = \frac{2\pi k}{5}$, $k = 0, 1, \ldots, 4$, i.e.

$$W(e^{j\frac{2\pi k}{5}}) = X(e^{j\frac{2\pi k}{5}}) \quad \text{for} \quad k = 0, 1, \ldots, 4.$$
Problem 6 (10%)  

A system for the discrete-time spectral analysis of continuous-time signals is shown in Figure 6-1.

\[ x_c(t) \xrightarrow{C/D} x[n] \xrightarrow{v[n]} 32\text{-point DFT} \xrightarrow{V[k]} \]

\[ T = 10^{-3} \text{ s} \]

\[ w[n] \] is a rectangular window of length 32:

\[ w[n] = \begin{cases} 
\frac{1}{32}, & 0 \leq n \leq 31 \\
0, & \text{otherwise} 
\end{cases} \]

Figure 6-1: Spectral analysis system.

Figure 6-2: Output \(|V[k]|\) in dB
Listed below are ten signals, at least one of which was the input \( x_c(t) \). Indicate which signal(s) could have been the input \( x_c(t) \) which produced the plot of \(|V[k]|\) shown in dB units in Figure 6-2. As always, provide reasoning for your choice(s).

\[
\begin{align*}
x_1(t) &= 1000 \cos(230\pi t) \\
x_2(t) &= 1000 \cos(115\pi t) \\
x_3(t) &= 10e^{j(460)\pi t} \\
x_4(t) &= 1000e^{j(230)\pi t} \\
x_5(t) &= 10e^{j(230)\pi t} \\
x_6(t) &= 1000e^{j(250)\pi t} \\
x_7(t) &= 10\cos(250\pi t) \\
x_8(t) &= 1000\cos(218.75\pi t) \\
x_9(t) &= 10e^{j(200)\pi t} \\
x_{10}(t) &= 1000e^{j(187.5)\pi t}
\end{align*}
\]

Work to be looked at and answer:
Problem 7  (12%)

$x[n]$ is a finite-length sequence of length 1024, i.e.
\[ x[n] = 0 \quad \text{for } n < 0, \ n > 1023. \]

The autocorrelation of $x[n]$ is defined as
\[ R_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]x[n+m], \]
and $X_N[k]$ is defined as the $N$-point DFT of $x[n]$, with $N \geq 1024$.

We are interested in computing $R_{xx}[m]$. A proposed procedure begins by first generating the $N$-point inverse DFT of $|X_N[k]|^2$ to obtain an $N$-point sequence $g_N[n]$, i.e.
\[ g_N[n] = \text{N-point IDFT} \left\{ |X_N[k]|^2 \right\}. \]

[6%] (a) Determine the minimum value of $N$ so that $R_{xx}[m]$ can be obtained from $g_N[n]$. Also specify how you would obtain $R_{xx}[m]$ from $g_N[n]$.

Work to be looked at and answer:
[6%] (b) Determine the minimum value of $N$ so that $R_{xx}[m]$ for $|m| \leq 10$ can be obtained from $g_N[n]$. Also specify how you would obtain these values from $g_N[n]$.

Work to be looked at and answer:
Problem 8  (8%) 

A system for examining the spectral content of a signal \( x[n] \) is shown in Figure 8-1. The filters \( h[n] \) in each channel are identical three-point non-causal FIR filters and have impulse response

\[
h[n] = h_0 \delta[n] + h_1 \delta[n + 1] + h_2 \delta[n + 2].
\]

The filter outputs are sampled at \( n = 0 \) to obtain the sequence \( y_k[0], k = 0, 1, 2, 3 \).

![Figure 8-1: Filter bank network.](image)

An alternative to the system in Figure 8-1 has been proposed using a 4-point DFT as shown in Figure 8-2.

![Figure 8-2: Alternative system.](image)
Determine $g[n]$ and $r[n]$ so that $P[k] = y_k[0]$.

Work to be looked at and answer:
Problem 9  (9%)

Consider the system shown in Figure 9-1. The subsystem from $x[n]$ to $y[n]$ is a causal, LTI system implementing the difference equation

$$y[n] = x[n] + ay[n - 1].$$

$x[n]$ is a finite length sequence of length 90, i.e.

$$x[n] = 0 \quad \text{for } n < 0 \text{ and } n > 89.$$

![Figure 9-1: System for calculating $y[M]$ from $x[n]$.](image)

Determine a choice for the complex constant $a$ and a choice for the sampling instant $M$ so that

$$y[M] = X(e^{j\omega})|_{\omega = 2\pi/60}$$

Work to be looked at and answer: