6.341 Discrete-Time Signal Processing
Fall 2005

MIDTERM EXAM
Tuesday, November 8, 2005

SOLUTIONS

Disclaimer: These are not meant to be full solutions, but rather an indication of how to approach each problem.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Grade</th>
<th>Points</th>
<th>Grader</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>/15</td>
<td></td>
</tr>
<tr>
<td>2 (a)</td>
<td></td>
<td>/12</td>
<td></td>
</tr>
<tr>
<td>2 (b)</td>
<td></td>
<td>/6</td>
<td></td>
</tr>
<tr>
<td>2 (c)</td>
<td></td>
<td>/12</td>
<td></td>
</tr>
<tr>
<td>3 (a)</td>
<td></td>
<td>/10</td>
<td></td>
</tr>
<tr>
<td>3 (b)</td>
<td></td>
<td>/10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>/15</td>
<td></td>
</tr>
<tr>
<td>5 (a)</td>
<td></td>
<td>/10</td>
<td></td>
</tr>
<tr>
<td>5 (b)</td>
<td></td>
<td>/10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>/100</td>
<td></td>
</tr>
</tbody>
</table>
Problem 1  (15%)

A stable system with system function $H(z)$ has the pole-zero diagram shown in Figure 1-1. It can be represented as the cascade of a stable minimum-phase system $H_{\text{min}}(z)$ and a stable all-pass system $H_{\text{ap}}(z)$.

![Figure 1-1: Pole-zero diagram for $H(z)$](image)

Determine a choice for $H_{\text{min}}(z)$ and $H_{\text{ap}}(z)$ (up to a scale factor) and draw their corresponding pole-zero plots. Indicate whether your decomposition is unique up to a scale factor.

We find a minimum-phase system $H_{\text{min}}(z)$ that has the same frequency response magnitude as $H(z)$ up to a scale factor. Poles and zeros that were outside the unit circle are moved to their conjugate reciprocal locations (3 to $\frac{1}{3}$, 4 to $\frac{1}{4}$, $\infty$ to 0).

$$H_{\text{min}}(z) = K_1 \frac{(1 - \frac{1}{4}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

There is no need to include an explicit $z$ term to account for the zero at the origin.

We now include all-pass terms in $H_{\text{ap}}(z)$ to move poles and zeros back to their original locations in $H(z)$. The term $\frac{z^{-1}-3}{1-3z^{-1}}$ moves the pole at $\frac{1}{3}$ back to 3, the term $z^{-1}$ moves the zero from 0 to $\infty$, and so on:

$$H_{\text{ap}}(z) = z^{-1} \left( \frac{z^{-1}-3}{1-3z^{-1}} \right) \left( \frac{z^{-1}-\frac{1}{4}}{1-\frac{1}{4}z^{-1}} \right)$$

The decomposition is unique up to a scale factor. We cannot include additional all-pass terms in $H_{\text{ap}}(z)$, since it is not possible for $H_{\text{min}}(z)$ to cancel the resulting poles and zeros outside the unit circle.
Figure 1-2: Pole-zero diagram for $H_{\text{min}}(z)$.

Figure 1-3: Pole-zero diagram for $H_{\text{ap}}(z)$. 
Problem 2  (30%) 

For parts (a) and (b) only, $X_c(j\Omega) = 0$ for $|\Omega| > 2\pi \times 10^3$ and $H(e^{j\omega})$ is as shown in Figure 2-2 (and of course periodically repeats).

(12%) (a) Determine the most general condition on $T$, if any, so that the overall continuous-time system from $x_c(t)$ to $y_c(t)$ is LTI.

We can simplify Figure 2-1 by replacing the expander, $H(e^{j\omega})$ and compressor with a single DT LTI system. The impulse response of the equivalent DT LTI system is $h[2n]$, where $h[n]$ is the inverse DTFT of $H(e^{j\omega})$. We saw this result in Problem Set 6 where we derived it by polyphase decomposition and subsequent simplification. The frequency response of the equivalent DT LTI system is $\frac{1}{2}H(e^{j\frac{\pi}{4}}) + \frac{1}{2}H(e^{j(\frac{\pi}{4} - \pi)})$.

Alternatively, we can reach the same conclusion by relating $Y_M(e^{j\omega})$ to $X_L(e^{j\omega})$:

$$Y_M(e^{j\omega}) = \frac{1}{2} \left[ H(e^{j\frac{\pi}{4}})X_L(e^{j2\frac{\pi}{4}}) + H(e^{j(\frac{\pi}{4} - \pi)})X_L(e^{j2(\frac{\pi}{4} - \pi)}) \right]$$

Both $X_L(\cdot)$ terms reduce to $X_L(e^{j\omega})$, the latter because of $2\pi$ periodicity, leaving behind the same equivalent frequency response.

The overall CT system will be LTI if there is no aliasing at the C/D converter, or if the DT filter rejects any frequency content contaminated by aliasing. Since the equivalent DT frequency response is 0 for $\frac{\pi}{2} < |\omega| \leq \pi$, the copy of $X_c(j\Omega)$ centred at $\omega = 0$ can extend to $\omega = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$ without passing any aliased components. Applying the $\omega = \Omega T$ mapping:

$$\omega_{max} = \Omega_{max} T = (2\pi \times 10^3) T < \frac{3\pi}{2}$$

and so we require that $T < \frac{3}{4} \times 10^{-3}$. 

---

```
\text{Figure 2-1:}

\text{Figure 2-2:}
```
(6%) (b) Sketch and clearly label the overall equivalent continuous-time frequency response $H_{\text{eff}}(j\Omega)$ that results when the condition determined in (a) holds.

\[ H_{\text{eff}}(j\Omega) \]
\[ \frac{A}{2} \]
\[ -\frac{\pi}{2T} \quad \frac{\pi}{2T} \]

(12%) (c) For this part only assume that $X_c(j\Omega)$ in Figure 2-1 is bandlimited to avoid aliasing, i.e. $X_c(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$. For a general sampling period $T$, we would like to choose the system $H(e^{j\omega})$ in Figure 2-1 so that the overall continuous-time system from $x_c(t)$ to $y_c(t)$ is LTI for any input $x_c(t)$ bandlimited as above.

Determine the most general conditions on $H(e^{j\omega})$, if any, so that the overall CT system is LTI. Assuming that these conditions hold, also specify in terms of $H(e^{j\omega})$ the overall equivalent continuous-time frequency response $H_{\text{eff}}(j\Omega)$.

The simplification in part (a) yields an equivalent DT LTI system between the C/D and D/C converters. Thus the overall system will be LTI if $x_c(t)$ is appropriately bandlimited to avoid aliasing, as assumed in this part. There are no conditions on $H(e^{j\omega})$.  

Problem 3  (20%) 

For this problem you may find the information on page 12 useful.

Consider the LTI system represented by the FIR lattice structure in Figure 3-1.

![Figure 3-1: LTI system diagram](image)

(10%) (a) Determine the system function from the input $x[n]$ to the output $v[n]$ (NOT $y[n]$).

The output $v[n]$ is taken after two stages, so we perform the lattice recursion up to order $p = 2$.

$$ k_1 = \frac{1}{2}, \quad k_2 = 3, \quad k_3 = 2 $$

$$ a_1^{(1)} = k_1 = -\frac{1}{2} $$

$$ \begin{bmatrix} a_1^{(2)} \\ a_2^{(2)} \end{bmatrix} = \begin{bmatrix} a_1^{(1)} \\ 0 \end{bmatrix} - k_2 \begin{bmatrix} a_1^{(1)} \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} $$

$$ \frac{V(z)}{X(z)} = 1 - z^{-1} - 3z^{-2} $$

Note the change of signs in going from $a_k^{(2)}$ to the system function.
(10%) (b) Let $H(z)$ be the system function from the input $x[n]$ to the output $y[n]$, and let $g[n]$ be the result of expanding the associated impulse response $h[n]$ by 2:

$$h[n] \uparrow 2 \rightarrow g[n]$$

The impulse response $g[n]$ defines a new system with system function $G(z)$.

We would like to implement $G(z)$ using an FIR lattice structure as defined by the figure on page 12. Determine the $k$-parameters necessary for an FIR lattice implementation of $G(z)$.

**Note:** You should think carefully before diving into a long calculation.

Since $g[n]$ is $h[n]$ expanded by 2, $G(z) = H(z^2)$. We replace $z$ by $z^2$ in Figure 3-1. We then expand each of the three sections as shown below:

The resulting flowgraph for $G(z)$ is in the form of a 6th-order FIR lattice. We read off the six $k$-parameters as:

$$k_2 = -\frac{1}{2}$$
$$k_4 = 3$$
$$k_6 = 2$$
$$k_p = 0, \quad p = 1, 3, 5$$
**Problem 4  (15%)**

We find in a treasure chest an even-symmetric FIR filter $h[n]$ of length $2L + 1$, *i.e.*

$$h[n] = 0 \text{ for } |n| > L,$$

$$h[n] = h[-n].$$

$H(e^{j\omega})$, the DTFT of $h[n]$, is plotted over $-\pi \leq \omega \leq \pi$ in Figure 4-1.

![Figure 4-1: Plot of $H(e^{j\omega})$ over $-\pi \leq \omega \leq \pi$.](image)
What can be inferred from Figure 4-1 about the possible range of values of $L$? Clearly explain the reason(s) for your answer. Do not make any assumptions about the design procedure that might have been used to obtain $h[n]$.

As discussed in Section 7.4 of OSB, $H(e^{j\omega})$ can be represented as an $L$th order polynomial in $\cos \omega$ since $h[n]$ is even-symmetric and finite-length. An $L$th order polynomial in $\cos \omega$ can have no more than $L + 1$ extrema on $[0, \pi]$. (An $L$th order polynomial in $x$ can have no more than $L - 1$ extrema in an open interval, but an $L$th order polynomial in $\cos \omega$ will always have additional extrema at $\omega = 0$ and $\omega = \pi$ for a total of $L + 1$ extrema.)

$H(e^{j\omega})$ has 6 extrema on $[0, \pi]$, so $6 \leq L + 1$ or $L \geq 5$. 
Problem 5  (20\%)

In Figure 5-1, $x[n]$ is a finite sequence of length 1024. The sequence $R[k]$ is obtained by taking the 1024-point DFT of $x[n]$ and compressing the result by 2.

(10\%) (a) Choose the most accurate statement for $r[n]$, the 512-point inverse DFT of $R[k]$. Justify your choice in a few concise sentences.

A. $r[n] = x[n]$, $0 \leq n \leq 511$
B. $r[n] = x[2n]$, $0 \leq n \leq 511$
C. $r[n] = x[n] + x[n + 512]$, $0 \leq n \leq 511$
D. $r[n] = x[n] + x[-n + 512]$, $0 \leq n \leq 511$
E. $r[n] = x[n] + x[1023 - n]$, $0 \leq n \leq 511$

In all cases $r[n] = 0$ outside $0 \leq n \leq 511$.

Answer: C

Compressing the 1024-point DFT $X[k]$ by 2 undersamples the DTFT $X(e^{j\omega})$. Undersampling in the frequency domain corresponds to aliasing in the time domain. In this specific case, the second half of $x[n]$ is folded onto the first half, as described by statement C.
(10%) (b) The sequence $Y[k]$ is obtained by expanding $R[k]$ by 2. Choose the most accurate statement for $y[n]$, the 1024-point inverse DFT of $Y[k]$. Justify your choice in a few concise sentences.

A. $y[n] = \begin{cases} \frac{1}{2} (x[n] + x[n + 512]), & 0 \leq n \leq 511 \\ \frac{1}{2} (x[n] + x[n - 512]), & 512 \leq n \leq 1023 \end{cases}$

B. $y[n] = \begin{cases} x[n], & 0 \leq n \leq 511 \\ x[n - 512], & 512 \leq n \leq 1023 \end{cases}$

C. $y[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

D. $y[n] = \begin{cases} x[2n], & 0 \leq n \leq 511 \\ x[2(n - 512)], & 512 \leq n \leq 1023 \end{cases}$

E. $y[n] = \frac{1}{2} (x[n] + x[1023 - n]), \ 0 \leq n \leq 1023$

In all cases $y[n] = 0$ outside $0 \leq n \leq 1023$.

**Answer:** A

We can first think about how expanding by 2 in the time domain affects the DFT. Expanding a time sequence $x[n]$ by 2 compresses the DTFT $X(e^{j\omega})$ by 2 in frequency. As a result, the $2N$-point DFT of the expanded sequence samples *two* periods of $X(e^{j\omega})$ and equals two copies of the $N$-point DFT $X[k]$.

By duality, expanding the DFT $R[k]$ by 2 corresponds to repeating $r[n]$ back-to-back, with an additional scaling by $\frac{1}{2}$. Thus statement A is correct.

Alternatively, $Y[k] = \frac{1}{2} (1 + (-1)^k) X[k]$. Modulating the DFT by $(-1)^k$ corresponds to a circular time shift of $N/2 = 512$.
You can use the blank part of this page as scratch paper but nothing on this page will be considered during grading.

\[ \begin{align*}
A_{p+1}(z) &= A_p(z) - k_{p+1}B_p(z) \\
B_p(z) &= z^{-(p+1)}A_p(1/z) \\
A_0(z) &= 1 \quad \text{and} \quad B_0(z) = z^{-1}
\end{align*} \]  

(1a)

(3a)

(2)

Then equations (8) become:

\[ \alpha_{p+1} = \begin{bmatrix} \alpha_p \\ \vdots \\ 0 \end{bmatrix} - k_{p+1} \begin{bmatrix} \tilde{\alpha}_p \\ \vdots \\ -1 \end{bmatrix} \]  

(9)

Reverse recursion:

\[ a_k^{(M-1)} = \frac{a_k^{(M)} + k_M a_{M-k}^{(M)}}{1 - k_M^2}, \quad k = 1, \ldots, M - 1 \]  

(11)

with \( k_M = a_M^{(M)} \), \( k_{M-1} = a_{M-1}^{(M-1)} \).