Problem 7.1
Let $N(t)$ be a Poisson counting process on $t \geq 0$ with rate $\lambda$. Let $\{y_i\}$ be a collection of statistically independent, identically-distributed random variables with mean and variance

$$
E[y_i] = m_y \\
\text{var } y_i = \sigma_y^2,
$$

respectively. Assume that the $\{y_i\}$ are statistically independent of the counting process $N(t)$ and define a new random process $y(t)$ on $t \geq 0$ via

$$
y(t) = \begin{cases}
0 & N(t) = 0 \\
\sum_{i=1}^{N(t)} y_i & N(t) > 0.
\end{cases}
$$

(a) Sketch a typical sample function of $N(t)$ and the associated typical sample function of $y(t)$.

(b) Use iterated expectation (condition on $N(t)$ in the inner average) to find $E[y(t)]$ and $E[y^2(t)]$ for $t \geq 0$.

(c) Prove that $y(t)$ is an independent-increments process on $t \geq 0$ and use this fact to find the covariance function $K_{yy}(t, s)$ for $t, s \geq 0$.

Problem 7.2 (practice)

(a) Let $x(t)$ be an independent increments process on $t \geq 0$ whose covariance function is $K_{xx}(t, s)$, for $t, s \geq 0$. Show that

$$
K_{xx}(t, s) = \text{var}[x(\min(t, s))] \quad \text{for } t, s \geq 0.
$$
(b) Suppose \( x(t) \) in part (a) has stationary increments. Show that

\[
m_x(t) = at + b, \quad \text{for } t \geq 0,
\]

and

\[
K_{xx}(t, s) = c \min(t, s) + d \quad \text{for } t, s \geq 0
\]

where \( a, b \) are constants and \( c, d \) are non-negative constants.

**Problem 7.3**
Let \( x[n] \) be a real-valued, discrete-time, zero-mean wide-sense stationary random process with correlation function \( R_{xx}[m] \) and spectrum \( S_{xx}(z) \).

(a) Suppose \( x[n] \) is the input to two real-valued linear time-invariant systems as depicted below, producing two new processes, \( y_1[n] \) and \( y_2[n] \). Find \( K_{y_1 y_2}[n] \) and \( S_{y_1 y_2}(z) \).

![Figure 1-1](image)

(b) Suppose next that \( H_1(e^{j\omega}) \) and \( H_2(e^{j\omega}) \) are non-overlapping frequency responses, i.e.,

\[
|H_1(e^{j\omega})| \cdot |H_2(e^{j\omega})| = 0, \quad \forall \omega.
\]

Show that in this case \( y_1[n] \) and \( y_2[m] \) are uncorrelated for all \( n \) and \( m \). Are \( y_1[n] \) and \( y_2[m] \) statistically independent (for all \( n \) and \( m \)) in the case that \( x[n] \) is a Gaussian random process? Explain.

**Problem 7.4**
Consider a stationary process \( y(t) \) satisfying the equation

\[
\frac{dy(t)}{dt} + 2y(t) = u(t)
\]

where \( u(t) \) is a zero-mean stationary process with covariance function

\[
K_{uu}(\tau) = \delta(\tau) + 4e^{-|\tau|}.
\]

Also assume that the transformation from \( u(t) \) to \( y(t) \) is linear, time-invariant, and stable.
(a) Determine $S_{yy}(s)$, $S_{uy}(s)$, $K_{yy}(\tau)$, and $K_{uy}(\tau)$.

(b) Find a stable shaping filter for $y(t)$, i.e., find the system function $H(s)$ of a stable, causal system so that if the input $w(t)$ to this system is white noise with spectral height 1 ($K_{ww}(\tau) = \delta(\tau)$), then the output has power spectral density $S_{yy}(s)$ found in part (a). Is your choice of $H(s)$ unique? Why or why not?

**Problem 7.5**
Consider the system depicted in Fig. 3-1.

![Figure 3-1](image)

The zero-mean, wide-sense stationary stochastic process $x(t)$ has covariance function

$$R_{xx}(\tau) = e^{-|\tau|}.$$  

We would like to find a linear time-invariant system with system function $H(s)$ so that the output $y(t)$ has covariance function

$$R_{yy}(\tau) = e^{-|\tau|}.$$  

(a) Find a suitable $H(s)$, such that $y(t)$ cannot be written in the form $y(t) = x(t-\tau)$ for some $\tau$. Is it unique? If your answer is yes, explain. If your answer is no, construct another suitable $H(s)$.

(b) Assume that we also want $R_{xy}(\tau)$ to have the form

$$R_{xy}(\tau) = Ke^{-2\tau}, \quad \tau > 0$$

for some constant $K$. Does a system function $H(s)$ exist that meets the specifications? If your answer is no, explain. If your answer is yes, calculate the corresponding $R_{xy}(\tau)$ for all $\tau$, and indicate whether it is uniquely specified.

**Problem 7.6**
Suppose $x[n]$ is a zero-mean, wide-sense stationary random process with

$$S_{xx}(e^{j\omega}) = 1.$$  

We observe $y[n] = h[n] * x[n]$ where $h[n] = 0$ for $n < 0$ and $n \geq 2$, $h[0] > 0$, and

$$S_{yy}(e^{j\omega}) = \frac{5}{4} - \cos(\omega).$$
(a) Sketch the pole-zero plot for $S_{yy}(z)$ and determine one possible impulse response $h[n]$ that is consistent with the information given.

(b) Show that your answer to part (a) is not unique by determining a second distinct $h[n]$ consistent with the information given.

(c) Suppose we also observe $w[n] = g[n] * x[n]$ where $g[n] = 0$ for $n < 0$ and $n \geq 2$, and $g[0] > 0$. If
\[
S_{ww}(e^{j\omega}) = \frac{5}{4} + \cos(\omega)
\]
and
\[
S_{yw}(e^{j\omega}) = \frac{1}{4} e^{j\omega} - e^{-j\omega},
\]
sketch pole-zero plots for $S_{ww}(z)$ and $S_{yw}(z)$, then determine one possible impulse response $h[n]$ that is consistent with the information given.

(d) Is your answer to (c) unique? If your answer is yes, explain. If your answer is no, construct a second distinct such $h[n]$.

**Problem 7.7 (practice)**

Let $x[n]$ be a discrete-time, zero-mean, wide-sense stationary Gaussian random process with unknown correlation function $R_{xx}[m]$.

(a) Define a new process $y[n]$ by the relation
\[
y[n] = x[n] x[n - m].
\]
Find the mean function and the covariance function of this new process.

(b) Define the time-average correlation function by
\[
\hat{R}_{xx}[m; N] = \frac{1}{N} \sum_{n=-(N-1)/2}^{(N-1)/2} x[n] x[n - m]
\]
where $N$ is an odd integer. Find the mean and the variance of $\hat{R}_{xx}[m; N]$.

(c) Suppose that the process $x[n]$ has a bounded spectral density, i.e.,
\[
S_{xx}(e^{j\omega}) \leq M < \infty, \quad \forall \omega.
\]
Show that $\hat{R}_{xx}[m; N]$ is a consistent estimator for $R_{xx}[m]$. 


Problem 7.8

We have determined a number of convenient properties related to \textit{linear} systems. In this problem, we consider a memoryless \textit{nonlinear} system and its properties. Consider a system whose output $y[n]$ is related to its input $x[n]$ by

$$y[n] = x^2[n], \quad \forall n.$$  

(a) If $x[n]$ is strict-sense stationary, must $y[n]$ be strict-sense stationary? Prove, or give a counterexample.

(b) If $x[n]$ is wide-sense stationary, must $y[n]$ be wide-sense stationary? Prove, or give a counterexample.

(c) If $x[n]$ is a Gaussian random process, must $y[n]$ be a Gaussian random process? Prove, or give a counterexample.

(d) (optional) If $x[n]$ has independent increments, must $y[n]$ have independent increments? Prove, or give a counterexample.

(e) (optional) If $x[n]$ is a Markov process, must $y[n]$ be a Markov process? Prove, or give a counterexample.

(f) Can $R_{yy}[n, m]$ be found in terms of the first and second moments of $x[n]$?

(g) Find an expression for $R_{yy}[n, m]$ when $x[n]$ is a zero-mean, stationary Gaussian process with a given $R_{xx}[n]$. 