COUNTING

Readings: [Bertsekas & Tsitsiklis], Section 1.6, and solved problems 57-58 (in 1st edition) or problems 61-62 (in 2nd edition). These notes only cover the part of the lecture that is not covered in [BT].

1 BANACH’S MATCHBOX PROBLEM

A mathematician starts the day with a full matchbox, containing \( n \) matches, in each pocket. Each time a match is needed, the mathematician reaches into a “random” pocket and takes a match out of the corresponding box. We are interested in the probability that the first time that the mathematician reaches into a pocket and finds an empty box, the other box contains exactly \( k \) matches.

Solution: The event of interest can happen in two ways:

(a) In the first \( 2n - k \) times, the mathematician reached \( n \) times into the right pocket, \( n - k \) times into the left pocket, and then, at time \( 2n - k + 1 \), into the right pocket.

(b) In the first \( 2n - k \) times, the mathematician reached \( n \) times into the left pocket, \( n - k \) times into the right pocket, and then, at time \( 2n - k + 1 \), into the left pocket.

Scenario (a) has probability

\[
\binom{2n-k}{n} \cdot \frac{1}{2^{2n-k}} \cdot \frac{1}{2}.
\]

Scenario (b) has the same probability. Thus, the overall probability is

\[
\binom{2n-k}{n} \cdot \frac{1}{2^{2n-k}}.
\]

2 MULTINOMIAL PROBABILITIES

Consider a sequence of \( n \) independent trials. At each trial, there are \( r \) possible results, \( a_1, a_2, \ldots, a_r \), and the \( i \)th result is obtained with probability \( p_i \). What is
the probability that in $n$ trials there were exactly $n_1$ results equal to $a_1$, $n_2$ results equal to $r_2$, etc., where the $n_i$ are given nonnegative integers that add to $n$?

Solution: Note that every possible outcome ($n$-long sequence of results) that involves $n_i$ results equal to $a_i$, for all $i$, has the same probability, $p_1^{n_1} \cdots p_r^{n_r}$. How many such sequences are there? Any such sequence corresponds to a partition of the set $\{1, \ldots, n\}$ of trials into subsets of sizes $n_1, \ldots, n_r$: the $i$th subset, of size $n_i$, indicates the trials at which the result was equal to $a_i$. Thus, using the formula for the number of partitions, the desired probability is equal to

$$\frac{n!}{n_1! \cdots n_r!} \cdot p_1^{n_1} \cdots p_r^{n_r}.$$