Problem 5.1
Here we shall derive the signal-to-noise ratio (SNR) optimality of squeezed states for quadrature measurements.

(a) We know that $\hat{a} = \hat{a}_1 + j\hat{a}_2$, with $\hat{a}_1$ and $\hat{a}_2$ Hermitian, and that $[\hat{a}_1, \hat{a}_2] = j/2$. Thus we have that,

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle (\hat{a}_1 - j\hat{a}_2)(\hat{a}_1 + j\hat{a}_2) \rangle.$$

Multiplying out and using the commutator we get,

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle \hat{a}_1^2 \rangle + \langle \hat{a}_2^2 \rangle - 1/2$$

Because mean-squared values equal variances plus squared-mean values we then have,

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle \Delta \hat{a}_1^2 \rangle + \langle \hat{a}_1^2 \rangle + \langle \Delta \hat{a}_2^2 \rangle + \langle \hat{a}_2^2 \rangle - 1/2.$$

Rearranging terms, and using the average photon number constraint, we find that,

$$\text{SNR} \leq \frac{N + 1/2 - \langle \Delta \hat{a}_2^2 \rangle - \langle \hat{a}_2 \rangle^2}{\langle \Delta \hat{a}_1^2 \rangle} - 1,$$

with equality if and only if $\langle \hat{a}^\dagger \hat{a} \rangle = N$. By making $\langle \hat{a}^\dagger \hat{a} \rangle = N$ and $\langle \hat{a}_2 \rangle = 0$, we can increase the SNR to,

$$\text{SNR} = \frac{N + 1/2 - \langle \Delta \hat{a}_2^2 \rangle}{\langle \Delta \hat{a}_1^2 \rangle} - 1.$$

(b) For fixed $N$ and $\langle \Delta \hat{a}_1^2 \rangle$, the SNR expression we have just derived is maximized by a minimum uncertainty state, i.e., one which satisfies $\langle \Delta \hat{a}_1^2 \rangle / \langle \Delta \hat{a}_2^2 \rangle = 1/16$, in which case

$$\text{SNR} = \frac{N + 1/2}{\langle \Delta \hat{a}_1^2 \rangle} - \frac{1}{(4\langle \Delta \hat{a}_1^2 \rangle)^2} - 1.$$

Defining $x = \langle \Delta \hat{a}_1^2 \rangle$, we can differentiate the preceding SNR expression to obtain,

$$\frac{d\text{SNR}}{dx} = -\frac{N + 1/2}{x^2} + \frac{1}{8x^3},$$

which has a unique root at $x = 1/8(N + 1/2)$. Differentiating a second time gives,

$$\frac{d^2\text{SNR}}{dx^2} = \frac{2(N + 1/2)}{x^3} - \frac{3}{8x^4},$$

which equals $-8^3(N + 1/2)^4 < 0$ at $x = 1/8(N + 1/2)$, so that the stationary point we have found is a maximum. The resulting optimal SNR value is then found, by substitution, to be $\text{SNR} = 4N(N + 1)$. 

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(c) For the squeezed state $|\beta; \mu, \nu\rangle$, we know that
\begin{align*}
\langle \hat{a}_1 \rangle &= \text{Re}(\mu^* \beta - \nu \beta^*), \\
\langle \Delta \hat{a}_1^2 \rangle &= \frac{|\mu - \nu|^2}{4}, \\
\langle \hat{a}^\dagger \hat{a} \rangle &= |\mu^* \beta - \nu \beta^*|^2 + |\nu|^2.
\end{align*}
Substituting in $\beta = \sqrt{N(N+1)}$, $\mu = (N+1)/\sqrt{2N+1}$, and $\nu = N/\sqrt{2N+1}$, we get
\begin{align*}
\langle \hat{a}_1 \rangle &= \sqrt{N(N+1)}/(2N+1), \\
\langle \Delta \hat{a}_1^2 \rangle &= 1/4(2N+1), \\
\langle \hat{a}^\dagger \hat{a} \rangle &= N(N+1)/(2N+1) + N^2/(2N+1) = N.
\end{align*}
This state therefore has $N$ photons on average, and its quadrature-measurement SNR equals the optimal value $4N(N+1)$.

(d) For the coherent state $|\sqrt{N}\rangle$ we have,
\begin{align*}
\langle \hat{a}_1 \rangle &= \text{Re}(\sqrt{N}) = \sqrt{N}, \\
\langle \Delta \hat{a}_1^2 \rangle &= 1/4, \\
\langle \hat{a}^\dagger \hat{a} \rangle &= N.
\end{align*}
This state therefore has $N$ photons on average, and its quadrature-measurement SNR equals $4N$. The optimal squeezed state has a larger quadrature-measurement SNR by a factor of $N+1$; for $N \gg 1$, this is an enormous SNR advantage.

Problem 5.2
Here we shall introduce the notion of normally-ordered forms.

(a) This is a straightforward exercise. We have that
\begin{align*}
\hat{F} &\equiv \hat{a}^\dagger \hat{a} = (\hat{a}^\dagger \hat{a} + [\hat{a}, \hat{a}^\dagger]) \hat{a} = \hat{a}^\dagger \hat{a}^2 + \hat{a}, \\
\hat{F} &\equiv \hat{a} \hat{a}^\dagger \hat{a} = \hat{a} (\hat{a}^\dagger - [\hat{a}, \hat{a}^\dagger]) = \hat{a}^2 \hat{a}^\dagger - \hat{a},
\end{align*}
whence
\[ F^{(n)}(\hat{a}^\dagger, \hat{a}) = \hat{a}^\dagger \hat{a}^2 + \hat{a}, \]
and
\[ F^{(a)}(\hat{a}, \hat{a}^\dagger) = \hat{a}^2 \hat{a}^\dagger - \hat{a}, \]
Because \( \hat{F} = F^{(n)}(\hat{a}^\dagger, \hat{a}) \) we have that,
\[
\langle \alpha | \hat{F} | \alpha \rangle = \langle \alpha | F^{(n)}(\hat{a}^\dagger, \hat{a}) | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a}^2 + \hat{a} | \alpha \rangle = \alpha^* \alpha^2 + \alpha,
\]
where the last equality follows from (repeated) use of the eigen relations,
\[
\hat{a} | \alpha \rangle = \alpha | \alpha \rangle \quad \text{and} \quad \langle \alpha | \hat{a}^\dagger = \langle \alpha | \alpha^*. \]

(b) Using the coherent-state identity resolution twice, we get
\[
\hat{G} = \hat{I}G\hat{I} = \int \int \frac{d^2 \alpha}{\pi} \frac{d^2 \beta}{\pi} \langle \alpha | \hat{G} | \beta \rangle \langle \beta | \rangle.
\]

(c) We know that \( \hat{F} = F^{(n)}(\hat{a}^\dagger, \hat{a}) \), thus
\[
\langle \alpha | \hat{F} | \beta \rangle = \langle \alpha | F^{(n)}(\hat{a}^\dagger, \hat{a}) | \beta \rangle = \langle \alpha | \hat{a}^\dagger \hat{a}^2 + \hat{a} | \beta \rangle = (\alpha^* \beta^2 + \beta) \langle \alpha | \beta \rangle.
\]
Using \( F^{(n)}(\alpha^*, \alpha) = \alpha^* \alpha^2 + \alpha \), from (a), with \( \alpha^* \) and \( \alpha \) treated as independent variables we now get,
\[
\langle \alpha | \hat{F} | \beta \rangle = F^{(n)}(\alpha^*, \beta) \langle \alpha | \beta \rangle = F^{(n)}(\alpha^*, \beta) e^{-|\alpha|^2 + |\beta|^2 + 1 + \alpha^* \beta},
\]
where the last result uses the coherent-state inner product that we have derived in a previous problem set.

(d) The density operator is an Hermitian operator whose eigenvalues form a probability distribution. Moreover, \( \langle \psi | \hat{\rho} | \psi \rangle \), for any unit-length ket \( | \psi \rangle \), is the probability that the oscillator will be found in state \( | \psi \rangle \). Because the coherent states are normalized to unit length, we have that \( \langle \alpha | \hat{\rho} | \alpha \rangle \geq 0 \). Because the coherent states resolve the identity and the trace of an operator can be computed by summing its matrix elements in this overcomplete basis, we have that,
\[
\int \frac{d^2 \alpha}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle = \text{tr}(\hat{\rho}) = 1,
\]
where the last equality was proven on a previous problem set. It follows that \( p(\alpha_1, \alpha_2) \equiv \rho^{(n)}(\alpha^*, \alpha)/\pi = \langle \alpha | \hat{\rho} | \alpha \rangle / \pi \) is a proper joint probability density for two real-valued random variables. We shall see in class that this density characterizes the measurement statistics of heterodyne detection, viz., a joint measurement of both quadratures of the oscillator.
Problem 5.3
Here we will introduce the three most important characteristic functions for quantum statistical analyses.

(a) We have that \([\hat{a}, \hat{a}^\dagger] = 1\), so that \([-\zeta^* \hat{a}, \zeta \hat{a}^\dagger] = -|\zeta|^2\). It follows that,

\[ [-\zeta^* \hat{a}, [-\zeta^* \hat{a}, \zeta \hat{a}^\dagger]] = 0, \]

and hence (from the identities given in the problem statement),

\[ e^{-\zeta^* \hat{a}^\dagger + \zeta \hat{a}} = e^{-\zeta^* \hat{a}} e^{|\zeta|^2/2} = e^{|\zeta|^2/2} e^{-\zeta^* \hat{a}} e^{-|\zeta|^2/2}. \]

Multiplying these equalities by \(\hat{\rho}\) and taking the trace we obtain the relations we were seeking:

\[ \chi^\rho_W(\zeta^* \zeta) = \chi^\rho_A(\zeta^* \zeta) e^{|\zeta|^2/2} = \chi^\rho_N(\zeta^* \zeta) e^{-|\zeta|^2/2}. \]

It is now easy to use these relations to find all three characteristic functions for the coherent-state density operator, \(\hat{\rho} = |\alpha\rangle \langle \alpha|\). We start with the normally-ordered characteristic function,

\[ \chi^\rho_N(\zeta^* \zeta) \equiv \text{tr} \left( \hat{\rho} e^{\zeta \hat{a}^\dagger - \zeta^* \hat{a}} \right) = \text{tr} \left( |\alpha\rangle \langle \alpha| e^{\zeta \hat{a}^\dagger - \zeta^* \hat{a}} \right) = \langle \alpha| e^{\zeta \hat{a}^\dagger - \zeta^* \hat{a}} |\alpha\rangle = e^{\zeta^* \alpha - \zeta \alpha}. \]

We then immediately obtain,

\[ \chi^\rho_W(\zeta^* \zeta) = \chi^\rho_N(\zeta^* \zeta) e^{-|\zeta|^2/2} = e^{\zeta^* \alpha - \zeta \alpha - |\zeta|^2/2}; \]

and

\[ \chi^\rho_A(\zeta^* \zeta) = \chi^\rho_W(\zeta^* \zeta) e^{-|\zeta|^2/2} = e^{\zeta^* \alpha - \zeta \alpha - |\zeta|^2}. \]

(b) We have that,

\[ \chi^\rho_A(\zeta^* \zeta) \equiv \text{tr} \left( \hat{\rho} e^{-\zeta^* \hat{a}^\dagger \zeta \hat{a}} \right). \]

Introducing,

\[ \hat{I} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha|, \]

in between the exponentials in the $\chi^\rho_A$ definition yields,

$$\chi^\rho_A(\zeta^*, \zeta) = \int \frac{d^2 \alpha}{\pi} \text{tr} \left( \hat{\rho} e^{-\zeta^* a} |\alpha\rangle \langle \alpha| e^{\zeta^* a^\dagger} \right)$$

$$= \int \frac{d^2 \alpha}{\pi} \text{tr} \left( \hat{\rho} |\alpha\rangle \langle \alpha| e^{-\zeta^* a + \zeta a^*} \right)$$

$$= \int \frac{d^2 \alpha}{\pi} |\alpha\rangle \langle \alpha| e^{-\zeta^* a + \zeta a^*}$$

$$= \int d\alpha_1 d\alpha_2 \rho^{(n)}(\alpha^*, \alpha) \frac{e^{2j\zeta_2 \alpha_1 - 2j\zeta_1 \alpha_2}}{\pi}$$

$$= \frac{\mathcal{F}[\rho^{(n)}(\alpha^*, \alpha)]}{\pi} \bigg| \begin{array}{c}
\int d\alpha_1 d\alpha_2 \rho^{(n)}(\alpha^*, \alpha) \frac{e^{2j\zeta_2 \alpha_1 - 2j\zeta_1 \alpha_2}}{\pi}
\end{array}$$

where $\mathcal{F}[x(t_1, t_2)]$ denotes the 2-D Fourier transform,

$$X(f_1, f_2) = \mathcal{F}[x(t_1, t_2)] = \int dt_1 dt_2 x(t_1, t_2) e^{-j2\pi(f_1 t_1 + f_2 t_2)}.$$ 

For future use, we note that the standard 2-D inverse Fourier transform,

$$x(t_1, t_2) = \mathcal{F}^{-1}[X(f_1, f_2)] = \int df_1 df_2 X(f_1, f_2) e^{j2\pi(f_1 t_1 + f_2 t_2)},$$

can be used to show that

$$\rho^{(n)}(\alpha^*, \alpha) = \int d\zeta_1 d\zeta_2 \chi^\rho_A(\zeta^*, \zeta) e^{-2j\zeta_2 \alpha_1 + 2j\zeta_1 \alpha_2}.$$ 

(c) All we need to do is to show that we can recover the diagonal elements in the coherent-state representation from

$$\hat{\rho} = \int \frac{d^2 \zeta}{\pi} \chi^\rho_A(\zeta^*, \zeta) e^{-\zeta^* a} e^{\zeta a^\dagger}.$$ 

This calculation is simple:

$$|\alpha\rangle \langle \alpha| \hat{\rho} |\alpha\rangle \rangle = \int \frac{d^2 \zeta}{\pi} \chi^\rho_A(\zeta^*, \zeta) e^{-\zeta^* a} e^{\zeta a^\dagger}$$

$$= \int \frac{d^2 \zeta}{\pi} \chi^\rho_A(\zeta^*, \zeta) e^{-\zeta^* a + \zeta a^*}$$

$$= \int \frac{d\zeta_1 d\zeta_2}{\pi} \chi^\rho_A(\zeta^*, \zeta) e^{-2j\zeta_2 \alpha_1 + 2j\zeta_1 \alpha_2},$$

which equals $\rho^{(n)}(\alpha^*, \alpha)$, as was to be shown, from the result of (b).
(d) If $\hat{\rho}$ has a proper $P$-representation, then

$$\chi^\rho_N(\zeta^*, \zeta) \equiv \text{tr}\left(\hat{\rho}e^{\zeta^*}\hat{a}^{\dagger}e^{-\zeta}\hat{a}\right)$$

$$= \int d^2 \alpha P(\alpha, \alpha^*)\text{tr}\left(|\alpha \rangle \langle \alpha |e^{\zeta^*}\hat{a}^{\dagger}e^{-\zeta}\hat{a}\right)$$

$$= \int d^2 \alpha P(\alpha, \alpha^*)e^{\zeta^\alpha - \zeta^*\alpha}$$

$$= \int d^2 \alpha P(\alpha, \alpha^*)e^{2j\zeta^\alpha_1 - 2j\zeta^\alpha_2},$$

again a 2-D Fourier transform relationship. Keeping track of the normalization constant (factors of $\pi$ in each Fourier dimension), we have that the inverse Fourier relationship is

$$P(\alpha, \alpha^*) = \int \frac{d^2 \zeta}{\pi^2} \chi^\rho_N(\zeta^*, \zeta)e^{-\zeta^\alpha + \zeta^*\alpha}.$$

(e) This part is trivial. We are told that the characteristic function for the classical outcome of the $\hat{a}_\theta$ measurement is

$$M_{\alpha}(jv) = \text{tr}\left(\hat{\rho}e^{jv\hat{a}_\theta}\right).$$

Substituting in the definition $\hat{a}_\theta = [\hat{a}e^{-j\theta} + \hat{a}^{\dagger}e^{j\theta}]/2$, we see that

$$M_{\alpha}(jv) = \text{tr}\left(\hat{\rho}e^{-jve^{-j\theta}/2}\hat{a} + jve^{j\theta}/2\hat{a}^{\dagger}\right) = \chi^\rho_W(-jve^{-j\theta}/2, jve^{j\theta}/2).$$

**Problem 5.4**

Here we shall that it is easy to calculate number-operator and quadrature-operator measurement statistics when the oscillator has a proper $P$-representation

(a) Suppose that the quantum harmonic oscillator is in the coherent state $|\alpha\rangle$ with classical probability density $p(\alpha_1, \alpha_2)$. If $\hat{O}$ is any observable, then the conditional probability that the outcome of this measurement will be the eigenvalue $o$, given that the oscillator is in the state $|\alpha\rangle$, is $|\langle o |\alpha\rangle|^2$. (Without appreciable loss of generality, we have assumed that the eigenspace associated with the eigenvalue $o$ is one-dimensional, and spanned by the unit-length eigenkets $\{|o\rangle\}$.) The unconditional probability that we get outcome $o$ is therefore,

$$\int \int d\alpha_1 d\alpha_2 p(\alpha_1, \alpha_2)|\langle o |\alpha\rangle|^2 = |\langle o | \left(\int d^2 \alpha p(\alpha_1, \alpha_2)|\alpha\rangle \langle \alpha |\right) |o\rangle.$$
But we know that the unconditional probability for the $\hat{O}$ measurement to yield outcome $o$ is given by, $\langle o | \hat{\rho} | o \rangle$, where $\hat{\rho}$ is the oscillator’s density operator. It follows, because $|o\rangle$ can be an arbitrary unit-length ket, that

$$\hat{\rho} = \int d^2\alpha p(\alpha_1, \alpha_2) |\alpha\rangle \langle \alpha|,$$

specifies the density operator in terms of the classical probability density for the state to be $|\alpha\rangle$. We are given that the density operator has a proper $P$-representation, i.e.,

$$\hat{\rho} = \int d^2\alpha P(\alpha, \alpha^*) |\alpha\rangle \langle \alpha|,$$

so it is clear that $p(\alpha_1, \alpha_2) = P(\alpha, \alpha^*)$ is the probability density that the state is $|\alpha\rangle$.

(b) When we are in the coherent state $|\alpha\rangle$ the $\hat{N}$-measurement has Poisson statistics,

$$\Pr(\hat{N} \text{ outcome } = n \mid \text{state is } |\alpha\rangle) = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}, \quad \text{for } n = 0, 1, 2, \ldots$$

Averaging over the proper $P$-representation—which specifies the probability density that the state will be $|\alpha\rangle$—then gives us the unconditional probability distribution:

$$\Pr(\hat{N} \text{ outcome } = n) = \int d^2\alpha P(\alpha, \alpha^*) \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}, \quad \text{for } n = 0, 1, 2, \ldots$$

Because the variance of the $\hat{N}$ measurement equals the mean of the conditional variance plus the variance of the conditional mean, and the variance of the conditional mean is non-negative, we have that

$$\langle \Delta \hat{N}^2 \rangle \geq \int d^2\alpha P(\alpha, \alpha^*) \text{var}(\hat{N} \text{ measurement } \mid \text{state is } |\alpha\rangle)$$

$$\geq \int d^2\alpha P(\alpha, \alpha^*) |\alpha|^2,$$

where the last equality uses the fact that the conditional distribution for the $\hat{N}$-measurement is Poisson with mean (and hence variance) $|\alpha|^2$. By a similar iterated expectation calculation we have that the mean of the $\hat{N}$-measurement equals the mean of its conditional mean, viz.,

$$\langle \hat{N} \rangle = \int d^2\alpha P(\alpha, \alpha^*) E(\hat{N} \text{ measurement } \mid \text{state is } |\alpha\rangle) = \int d^2\alpha P(\alpha, \alpha^*) |\alpha|^2,$$

completing the proof that states with proper $P$-representations satisfy,

$$\langle \Delta \hat{N}^2 \rangle \geq \langle \hat{N} \rangle.$$ 

Note that the number state $|n\rangle$ has $\langle \hat{N} \rangle = n$ and $\langle \Delta \hat{N}^2 \rangle = 0$, and so the density operator $\hat{\rho} = |n\rangle \langle n|$ does not have a proper $P$-representation for $n > 0$. 

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(c) When we are in the coherent state $|\alpha\rangle$, the probability density for the outcome of the $\hat{a}_1$ quadrature measurement is Gaussian with mean $\alpha_1 = \text{Re}(\alpha)$ and variance $1/4$. Thus the unconditional density function for this measurement outcome to be $a_1$, when the density operator has a proper $P$-representation is,

$$p(\hat{a}_1 \text{ outcome } = a_1) = \int d^2 \alpha P(\alpha, \alpha^*) \frac{\exp[-2(a_1 - \alpha_1)^2]}{\sqrt{\pi/2}}.$$ 

Via the same iterated expectation approach used in (b), we know that the variance of the $\hat{a}_1$ measurement equals or exceeds the mean of the conditional variance, i.e.,

$$\langle \Delta \hat{a}_1^2 \rangle \geq \int d^2 \alpha P(\alpha, \alpha^*) \text{var}(\hat{a}_1 \text{ measurement } | \text{ state is } |\alpha\rangle)$$

$$= \int d^2 \alpha P(\alpha, \alpha^*) \frac{1}{4} = \frac{1}{4}.$$ 

Note that the squeezed state $|\beta; \mu, \nu\rangle$ with $\mu, \nu > 0$ has $\langle \Delta \hat{a}_1^2 \rangle = (\mu - \nu)^2 / 4 < 1/4$, and so the density operator $\hat{\rho} = |\beta; \mu, \nu\rangle\langle \beta; \mu, \nu|$ for $\mu, \nu > 0$ does not have a proper $P$-representation.