6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

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Problem 1

A

Question: Find a suitable image current to find the magnetic field for $z > 0$. Does the direction of the image current surprise you?

Solution:

![Image showing magnetic field](imagecurrent)

Figure 1: Figure showing image line. (Image by MIT OpenCourseWare).

B

Question: What is the magnetic field magnitude and direction for $z > 0$?

Solution:

$$H_\phi = \frac{I}{2\pi r} \quad \text{for } z > 0$$
C

Question: What is the surface current magnitude and direction on the $z = 0$ surface of the conducting plane?

Solution:
\[ \pi \times \mathbf{H}(z = 0) = \mathbf{\bar{I}}_z \times H_\phi(z = 0) \mathbf{\bar{I}}_\phi = -H_\phi \mathbf{\bar{K}} \]
\[ K_r = -H_\phi(z = 0) = -\frac{I}{2\pi r} \]

Problem 2

A

Question: What is the electric field for $a \leq r \leq b$?

Solution:
\[ \nabla \cdot \mathbf{J} = \nabla \cdot [\sigma(r)\mathbf{E}] = 0 \quad (\mathbf{E} = E_r(r)\mathbf{\bar{r}}) \]
\[ \nabla \cdot [\sigma(r)\mathbf{E}] = \frac{1}{r} \frac{\partial}{\partial r} (r \sigma(r) E_r(r)) = 0 \]
\[ \sigma(r) = \frac{\sigma_0 r}{a} \]
\[ r \sigma(r) E_r(r) = C(\text{Constant}) = \frac{r^2 \sigma_0}{a} E_r(r) \]
\[ E_r(r) = \frac{Ca}{\sigma_0 r^2} \]
\[ v = \int_{r=a}^{b} E_r(r) dr = \int_{r=a}^{b} \frac{Ca}{\sigma_0 r^2} dr = -\frac{Ca}{\sigma_0} \left[ \frac{1}{r} \right]_{r=a}^{b} = -\frac{Ca}{\sigma_0} \left( \frac{1}{b} - \frac{1}{a} \right) \]
\[ C = \frac{\sigma_0 v}{1 - \frac{a}{b}} \Rightarrow E_r(r) = \frac{\sigma_0 va}{\sigma_0 r^2 (1 - \frac{a}{b})} = \frac{va}{r^2 (1 - \frac{a}{b})} \]

B

Question: What are the surface charge densities at $r = a$ and $r = b$?

Solution:
\[ \sigma_s(r = a) = \varepsilon E_r(r = a) = \frac{\varepsilon va}{a^2 (1 - \frac{b}{a})} = \frac{\varepsilon v}{a (1 - \frac{a}{b})} \]
\[ \sigma_s(r = b) = -\varepsilon E_r(r = b) = -\frac{\varepsilon va}{b^2 (1 - \frac{b}{a})} \]
C

**Question:** What is the volume charge density for $a \leq r \leq b$?

**Solution:**

$$
\rho = \varepsilon \nabla \cdot E = \frac{\varepsilon}{r} \frac{\partial}{\partial r} (rE_r) = \frac{\varepsilon}{r} \frac{\partial}{\partial r} \left( \frac{va}{r \left(1 - \frac{a}{b}\right)} \right) \\
= -\frac{\varepsilon va}{r^3 \left(1 - \frac{a}{b}\right)}
$$

D

**Question:** What is the total charge in the system?

**Solution:**

$$
L \int_a^b \rho 2\pi r dr = -\frac{2\pi \varepsilon vaL}{(1 - \frac{a}{b})} \int_a^b \frac{1}{r^2} dr = \frac{2\pi \varepsilon vaL}{(1 - \frac{a}{b})} \left. \frac{1}{r} \right|_a^b \\
= \frac{2\pi \varepsilon vaL}{(1 - \frac{a}{b})} \left( \frac{1}{b} - \frac{1}{a} \right) \\
= -2\pi \varepsilon vL
$$

$$
Q_T = \left[ 2\pi a \sigma_s(r = a) + 2\pi b \sigma_s(r = b) + \int_a^b \rho 2\pi r dr \right] L \\
= \frac{2\pi a \varepsilon v}{(1 - \frac{a}{b})} \left[ \frac{1}{a} - \frac{1}{b} + \frac{1}{b} - \frac{1}{a} \right] L \\
= 0
$$

E

**Question:** What is the resistance between the cylindrical electrodes?

**Solution:**

$$
i = \sigma(r)E_r(r)2\pi r L = \frac{\sigma_0 f}{\varepsilon} 2\pi f L \frac{v f}{\varepsilon} \left(1 - \frac{a}{b}\right) \\
= \frac{\sigma_0 2\pi L v}{(1 - \frac{a}{b})}
$$

$$
R = \frac{v}{i} = \frac{(1 - \frac{a}{b})}{\sigma_0 2\pi L}
$$
Problem 3

A

**Question:** There is no volume charge for \( 0 < r < R \) and \( r > R \) and \( \Phi(r = \infty, \theta) = 0 \). Laplace’s equation for the scalar electric potential in spherical coordinates is:

\[
\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0
\]

Guess a solution to Laplace’s equation of the form \( \Phi(r, \theta) = A r^p \cos \theta \) and find all allowed values of \( p \).

**Solution:**

\[
\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0
\]

\[
\Phi(r, \theta) = A r^p \cos \theta
\]

\[
\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 A r^{p-1} \cos \theta \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \left( -A r^p \sin \theta \right) \right) = 0
\]

\[
0 = A p \cos \theta \frac{\partial}{\partial r} \left( r^{p+1} \right) - \frac{1}{\sin \theta} A r^p \frac{\partial}{\partial \theta} \left( \sin^2 \theta \right)
\]

\[
= A r^p \cos \theta \left( p (p + 1) \right) - \frac{A r^p}{\sin \theta} 2 \sin \theta \cos \theta
\]

\[
= A r^p \cos \theta \left( p (p + 1) - 2 \right) = 0
\]

\[
p^2 + p - 2 = (p + 2)(p - 1) = 0 \Rightarrow p = 1, p = -2
\]

\[
\Phi_1(r, \theta) = A r \cos \theta, \Phi_2(r, \theta) = \frac{A \cos \theta}{r^2}
\]

B

**Question:** Which of your scalar electric potential solutions in part (a) are finite at \( r = 0 \)?

**Solution:**

\[
\Phi_1(r, \theta) = A r \cos \theta
\]

C

**Question:** Which of your solutions in part (a) have zero potential at \( r = \infty \)?

**Solution:**

\[
\Phi_2(r, \theta) = \frac{A \cos \theta}{r^2}
\]
D

Question: Using the results of parts (b) and (c) find the scalar electric potential solutions for \(0 \leq r \leq R\) and \(r \geq R\) that satisfy the boundary condition \(\Phi(r = R, \theta) = V_0 \cos \theta\).

Solution:

\[
\Phi(r, \theta) = \begin{cases} 
Ar \cos \theta & 0 \leq r \leq R \\
\frac{B}{r} \cos \theta & r \geq R
\end{cases}
\]

\[
\Phi(r = R, \theta) = V_0 \cos \theta = AR \cos \theta = \frac{B}{R^2} \cos \theta
\]

\[
A = \frac{V_0}{R}, \quad B = V_0 R^2
\]

\[
\Phi(r, \theta) = \begin{cases} 
\frac{V_0 r}{R} \cos \theta & 0 \leq r \leq R \\
\frac{V_0 R^2}{r^2} \cos \theta & r \geq R
\end{cases}
\]

E

Question: Find the electric field in the regions \(0 \leq r < R\) and \(r > R\).

Hint:

\[
\vec{E} = -\nabla \Phi = - \left[ \frac{\partial \Phi}{\partial r} \vec{r} + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \vec{\theta} \right]
\]

Solution:

\[
\vec{E} = - \left[ \frac{\partial \Phi}{\partial r} \vec{r} + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \vec{\theta} \right]
\]

\(0 \leq r < R\)

\[
\vec{E} = - \frac{V_0}{R} \left[ \cos \theta \vec{r} - \sin \theta \vec{\theta} \right]
\]

\(r > R\)

\[
\vec{E} = -V_0 R^2 \left[ -\frac{2}{r^3} \cos \theta \vec{r} - \frac{\sin \theta}{r^3} \vec{\theta} \right]
\]

\[
= \frac{V_0 R^2}{r^3} \left( 2 \cos \theta \vec{r} + \sin \theta \vec{\theta} \right)
\]

F

Question: What is the surface charge distribution on the \(r = R\) interface?

Solution:

\[
\sigma_s(r = R, \theta) = \varepsilon_0 E_r(r = R, \theta) - \varepsilon E_r(r = R, \theta)
\]

\[
= \frac{\varepsilon_0 V_0}{R} 2 \cos \theta + \varepsilon \frac{V_0}{R} \cos \theta
\]

\[
= \frac{V_0}{R} (\varepsilon + 2\varepsilon_0) \cos \theta
\]